Mathematical Relativity

Homework 1

Due on February 24

1. **Twin paradox:** Two twins, Alice and Bob, part on their 20^{th} birthday. While Alice remains on Earth (which is an inertial frame to a very good approximation), Bob departs at 80% of the speed of light towards Planet X, 8 light-years away from Earth. Therefore Bob reaches his destination 10 years later (as measured on the Earth's frame). After a short stay, he returns to Earth, again at 80% of the speed of light. Consequently Alice is 40 years old when she sees Bob again.

   (a) How old is Bob when they meet again?

   (b) How can the asymmetry in the twins' ages be explained? Notice that from Bob's point of view he is at rest in his spaceship and it is the Earth which moves away and then back again.

   (c) Imagine that each twin watches the other trough a very powerful telescope. What do they see? In particular, how much time do they experience as they see one year elapse for their twin?

2. A particularly simple matter model is that of a smooth **massless scalar field** $\phi : M \to \mathbb{R}$, whose energy-momentum tensor is

$$T_{\mu\nu} = \nabla_\mu \phi \nabla_\nu \phi - \frac{1}{2}(\nabla_\alpha \phi \nabla^\alpha \phi) g_{\mu\nu}.$$

   Show that if the Lorentzian manifold (M, g) satisfies the Einstein equations with this matter model then ϕ satisfies the **wave equation**

$$\Box \phi = 0 \Leftrightarrow \nabla^\mu \nabla_\mu \phi = 0.$$