Mathematical Quantum Mechanics

Homework 1

Due on September 17

1. The Schrödinger equation for an electron orbiting a proton in a hydrogen atom is

$$i\hbar\frac{\partial\psi}{\partial t}(\mathbf{x},t) = -\frac{\hbar^2}{2m}\Delta\psi(\mathbf{x},t) - \frac{e^2}{4\pi\varepsilon_0|\mathbf{x}|}\psi(\mathbf{x},t),$$

where m and $-e$ are the electron's mass and electric charge.

(a) Show that

$$\psi(\mathbf{x},t) = \frac{1}{\sqrt{\pi a_0^3}}e^{-|\mathbf{x}|/a_0}e^{-iEt/\hbar}$$

is a solution to this equation, where

$$a_0 = \frac{4\pi\varepsilon_0\hbar^2}{me^2}$$

is Bohr's radius and

$$E = -\frac{e^2}{8\pi\varepsilon_0 a_0}$$

is the energy for the lowest circular orbit in Bohr's atom.

(b) Show that ψ is normalized, that is, show that

$$\|\psi(\cdot,t)\|^2_{L^2(\mathbb{R}^3)} \equiv \int_{\mathbb{R}^3}|\psi(\mathbf{x},t)|^2 d^3x = 1$$

for all $t \in \mathbb{R}$.

(c) Compute the expected value of the distance to the origin (that is, of $|\mathbf{x}|$) given the probability distribution $|\psi(\mathbf{x},t)|^2$.

Remark: As we will see in due time, ψ represents the ground state (that is, the state with minimum energy) of the electron in the hydrogen atom.