Geometric Mechanics

Homework 4

Due on January 24

1. In this exercise we study in detail the timelike and null geodesics of the Schwarzschild spacetime. By symmetry, it suffices to study the geodesics of the totally geodesic 3-dimensional Lorentzian submanifold $\theta = \frac{\pi}{2}$ (**equatorial plane**), whose metric is

$$g = -\left(1 - \frac{2m}{r}\right) dt \otimes dt + \left(1 - \frac{2m}{r}\right)^{-1} dr \otimes dr + r^2 d\varphi \otimes d\varphi.$$

   (a) Show that the equations for a future-directed geodesic (parameterized by proper time if timelike) can be written as

$$\begin{cases} \dot{r}^2 = E^2 - \left(\sigma + \frac{L^2}{r^2}\right)\left(1 - \frac{2m}{r}\right) \\ \left(1 - \frac{2m}{r}\right)\dot{t} = E \\ r^2\dot{\varphi} = L \end{cases}$$

   where $E > 0$ and L are integration constants, $\sigma = 1$ for timelike geodesics and $\sigma = 0$ for null geodesics.

   (b) Show that if $L \neq 0$ then $u = \frac{1}{r}$ satisfies

$$\frac{d^2u}{d\varphi^2} + u = \frac{m\sigma}{L^2} + 3mu^2.$$

   (c) For situations where relativistic corrections are small one has $mu \ll 1$, and hence the approximate equation

$$\frac{d^2u}{d\varphi^2} + u = \frac{m}{L^2}$$

   holds for timelike geodesics. Show that the solution to this equation is the conic section given in polar coordinates by

$$u = \frac{m}{L^2}(1 + \varepsilon\cos(\varphi - \varphi_0)),$$

   where the integration constants $\varepsilon \geq 0$ and φ_0 are the eccentricity and the argument of the pericenter.

   (d) Show that for $\varepsilon \ll 1$ this approximate solution satisfies

$$u^2 \simeq \frac{2m}{L^2}u - \frac{m^2}{L^4}.$$

   Argue that timelike geodesics close to circular orbits where relativistic corrections are small yield approximate solutions of the equation

$$\frac{d^2u}{d\varphi^2} + \left(1 - \frac{6m^2}{L^2}\right)u = \frac{m}{L^2}\left(1 - \frac{3m^2}{L^2}\right),$$

and hence the pericenter advances by approximately

$$\frac{6\pi m}{r}$$

radians per revolution.

(**Remark:** The first success of general relativity was using this effect to explain the anomalous **precession of Mercury's perihelion** – 43 arcseconds per century).

(e) Show that if one neglects relativistic corrections then null geodesics satisfy

$$\frac{d^2u}{d\varphi^2} + u = 0.$$

Show that the solution to this equation is the equation for a straight line in polar coordinates,

$$u = \frac{1}{b}\sin(\varphi - \varphi_0),$$

where the integration constants $b > 0$ and φ_0 are the **impact parameter** (distance of closest approach to the center) and the angle between the line and the x-axis.

(f) Assume that $mu \ll 1$. Let us include relativistic corrections by looking for approximate solutions of the form

$$u = \frac{1}{b}\left(\sin\varphi + \frac{m}{b}v\right)$$

(where we take $\varphi_0 = 0$ for simplicity). Show that v is an approximate solution of the equation

$$\frac{d^2v}{d\varphi^2} + v = 3\sin^2\varphi,$$

and hence u is approximately given by

$$u = \frac{1}{b}\left(\sin\varphi + \frac{m}{b}\left(\frac{3}{2} + \frac{1}{2}\cos(2\varphi) + \alpha\cos\varphi + \beta\sin\varphi\right)\right),$$

where α and β are integration constants.

(g) Show that for the incoming part of the null geodesic ($\varphi \simeq 0$) one approximately has

$$u = 0 \Leftrightarrow \varphi = -\frac{m}{b}(2 + \alpha).$$

Similarly, show that for the outgoing part of the null geodesic ($\varphi \simeq \pi$) one approximately has

$$u = 0 \Leftrightarrow \varphi = \pi + \frac{m}{b}(2 - \alpha).$$

Conclude that φ varies by approximately

$$\Delta\varphi = \pi + \frac{4m}{b}$$

radians along its path, and hence the null geodesic is deflected towards the center by approximately

$$\frac{4m}{b}$$

radians.

(**Remark:** The measurement of this **deflection of light** by the Sun – 1.75 arcseconds – was the first experimental confirmation of general relativity, and made Einstein a world celebrity overnight).

2. Consider two galaxies in a FLRW model

$$g = -dt \otimes dt + a^2(t) \left(\frac{1}{1-kr^2} dr \otimes dr + r^2 d\theta \otimes d\theta + r^2 \sin^2 \theta d\varphi \otimes d\varphi \right),$$

whose spatial locations can be assumed to be $r = 0$ and $(r, \theta, \varphi) = (r_1, \theta_1, \varphi_1)$.

(a) Show that the family (reparameterized) null geodesics connecting the first galaxy to the second galaxy can be written as

$$(t, r, \theta, \varphi) = (t(r, t_0), r, \theta_1, \varphi_1) \qquad (0 < r < r_1),$$

where $t(r, t_0)$ is the solution of

$$\begin{cases} \dfrac{dt}{dr} = \dfrac{a(t)}{\sqrt{1-kr^2}} \\ \\ t(0, t_0) = t_0 \end{cases}.$$

(b) Prove that $\dfrac{\partial t}{\partial t_0}(r_1, t_0) = \dfrac{a(t_1)}{a(t_0)}$, where $t_1 = t(r_1, t_0)$.

(c) The **redshift** of the light propagating from the first galaxy to the second galaxy is defined as

$$z = \frac{\partial t}{\partial t_0}(r_1, t_0) - 1 = \frac{a(t_1)}{a(t_0)} - 1.$$

This light is spread over a sphere of radius $R = a(t_1) r_1$, and so its brightness is inversely proportional to R^2. Compute R as a function of z and t_1 for the following FLRW models:

(i) **Milne universe**, for which $k = -1$ and $a(t) = t$;

(ii) **Flat de Sitter universe**, for which $k = 0$ and $a(t) = e^{Ht}$;

(iii) **Einstein-de Sitter universe**, for which $k = 0$ and $a(t) = (t/t_1)^{2/3}$.

(**Remark:** The brightness of distant galaxies is further reduced by a factor of $(1+z)^2$, since each photon has frequency, hence energy, $(1+z)$ times smaller at reception, and the rate of detection of photons is $(1+z)$ times smaller than the rate of emission; with this correction, R can be deduced from the observed brightness for galaxies of known luminosity, and the correct FLRW model can be chosen as the one whose curve $R = R(z)$ best fits observations).