# Geometric Mechanics

## Homework 3

*Due on January 5*

1. Let $(M, \langle \cdot, \cdot \rangle)$ be a Riemannian manifold and $p, q \in M$. A curve of minimal length connecting $p$ to $q$, with nonvanishing derivative, must be a critical point of the action determined by the Lagrangian $L : TM \setminus Z \to \mathbb{R}$ given by

$$L(v) = \langle v, v \rangle^{\frac{1}{2}}$$

($Z \subset TM$ is the zero section).

   (a) Show that such a curve is a reparameterized geodesic. (**Hint:** Write $L(v) = (2K(v))^{\frac{1}{2}}$).

   (b) Compute the Hamiltonian function $H : TM \setminus Z \to \mathbb{R}$.

2. The motion of a particle with mass $m > 0$ and charge $e \in \mathbb{R}$ in a stationary electromagnetic field is determined by the Lagrangian $L : T\mathbb{R}^3 \to \mathbb{R}$ given by

$$L = \frac{1}{2} m \langle v, v \rangle + e \langle A, v \rangle - e\,\Phi,$$

where $\langle \cdot, \cdot \rangle$ is the Euclidean inner product, $\Phi \in C^\infty(\mathbb{R}^3)$ is the **electric potential** and $A \in \mathfrak{X}(\mathbb{R}^3)$ is the **magnetic vector potential**.

   (a) Show that the equations of motion are

$$m\ddot{x} = eE + e\dot{x} \times B,$$

   where $E = -\operatorname{grad} \Phi$ is the **electric field** and $B = \operatorname{curl} A$ is the **magnetic field**. (**Remark:** In particular, adding a gradient to $A$ changes the Lagrangian but not the Euler-Lagrange equations).

   (b) Write an expression for the Hamiltonian function and use the equations of motion to check that it is constant along any motion.

   (c) Use Noether's theorem to show that if there exists a vector $u \in \mathbb{R}^3$ such that $u \cdot \Phi = 0$ and $\nabla_u A = 0$ then the quantity

$$J = \langle m\dot{x} + eA, u \rangle$$

   is conserved along the motions of the particle.

3. Recall that the Lagrange top is the mechanical system determined by the Lagrangian function $L : TSO(3) \to \mathbb{R}$ given in local coordinates by

$$L = \frac{I_1}{2}\left(\left(v^\theta\right)^2 + \left(v^\varphi\right)^2 \sin^2\theta\right) + \frac{I_3}{2}\left(v^\psi + v^\varphi \cos\theta\right)^2 - Mgl\cos\theta,$$

where $(\theta, \varphi, \psi)$ are the Euler angles, $M$ is the top's mass and $l$ is the distance from the fixed point to the center of mass.

(a) Compute the Legendre transformation, show that $L$ is hyper-regular and write an expression in local coordinates for the Hamiltonian $H : T^*SO(3) \to \mathbb{R}$.

(b) Prove that $H$ is completely integrable.

(c) Find all solutions with constant $\theta$, $\dot{\varphi}$ and $\dot{\psi}$, and argue that they are stable for $|\dot{\varphi}| \ll |\dot{\psi}|$ if $|\dot{\psi}|$ is large enough.

4. Consider the sequence formed by the first digit of the decimal expansion of each of the integers $2^n$ for $n \in \mathbb{N}_0$:

$$1, 2, 4, 8, 1, 3, 6, 1, 2, 5, 1, 2, 4, 8, 1, 3, 6, 1, 2, 5, \ldots$$

The purpose of this exercise is to answer the following question: is there a 7 in this sequence?

(a) Show that if $\nu \in \mathbb{R} \setminus \mathbb{Q}$ then

$$\lim_{n\to+\infty} \frac{1}{n+1} \sum_{k=0}^{n} e^{2\pi i \nu k} = 0.$$

(b) Prove the following discrete version of the Birkhoff Ergodic Theorem: if a smooth function $f : \mathbb{R} \to \mathbb{R}$ is periodic with period $1$ and $\nu \in \mathbb{R} \setminus \mathbb{Q}$ then for all $x \in \mathbb{R}$

$$\lim_{n\to+\infty} \frac{1}{n+1} \sum_{k=0}^{n} f(x + \nu k) = \int_0^1 f(x)dx.$$

(c) Show that $\log 2$ is an irrational multiple of $\log 10$.

(d) Is there a 7 in the sequence above?