Geometric Mechanics

Homework 2

Due on November 24

1. Let $(M, \langle\cdot,\cdot\rangle, \mathcal{F})$ be a mechanical system, with $(x^1, \ldots, x^n)$ local coordinates on M and $(x^1, \ldots, x^n, v^1, \ldots, v^n)$ the corresponding local coordinates on TM. Show that

$$\mu\left(\frac{D\dot{c}}{dt}(t)\right) = \sum_{i=1}^{n}\left[\frac{d}{dt}\left(\frac{\partial K}{\partial v^i}(x(t), \dot{x}(t))\right) - \frac{\partial K}{\partial x^i}(x(t), \dot{x}(t))\right] dx^i$$

for any curve $c : \mathbb{R} \to M$, where $x(t) = x(c(t))$.

2. The **spherical pendulum** of length l is the mechanical system defined by a particle of mass $m > 0$ moving in $\mathbb{R}^3$ subject to a constant gravitational acceleration g and the holonomic constraint

$$N = \{(x, y, z) \in \mathbb{R}^3 \mid x^2 + y^2 + z^2 = l^2\}$$

(assuming a perfect reaction force).

   (a) Write the equations of motion for the spherical pendulum using spherical coordinates.

   (b) Which parallels of N are (images of) motions of the system?

3. Recall that the **Lagrange top** is the mechanical system in $SO(3)$ whose kinetic energy in the local coordinates $(\theta, \varphi, \psi, v^\theta, v^\varphi, v^\psi)$ of $TSO(3)$ associated to the Euler angles is

$$K = \frac{I_1}{2}\left(\left(v^\theta\right)^2 + \left(v^\varphi\right)^2 \sin^2\theta\right) + \frac{I_3}{2}\left(v^\psi + v^\varphi\cos\theta\right)^2,$$

and whose potential energy is

$$U = Mgl\cos\theta.$$

Show that there exist solutions of this mechanical system such that θ, $\dot{\varphi}$ and $\dot{\psi}$ are constant, which in the limit $|\dot{\varphi}| \ll |\dot{\psi}|$ (**fast top**) satisfy

$$\dot{\varphi} \simeq \frac{Mgl}{I_3\dot{\psi}}.$$

4. Recall that our model for an ice skate is given by the non-holonomic constraint Σ determined on $\mathbb{R}^2 \times S^1$ by the kernel of the 1-form $\omega = -\sin\theta dx + \cos\theta dy$.

   (a) Show that the ice skate can access all points in the configuration space: given two points $p, q \in \mathbb{R}^2 \times S^1$, there exists a piecewise smooth curve $c : [0,1] \to \mathbb{R}^2 \times S^1$, compatible with Σ, such that $c(0) = p$ and $c(1) = q$. Why does this show that Σ is non-integrable?

   (b) Assuming that the kinetic energy of the skate is

$$K = \frac{M}{2}\left((v^x)^2 + (v^y)^2\right) + \frac{I}{2}\left(v^\theta\right)^2$$

   and that the reaction force is perfect, show that the skate moves with constant speed along straight lines or circles. What is the physical interpretation of the reaction force?