Geometric Mechanics

Homework 1

Due on October 13

1. The equation of motion for a particle of mass m moving on the real line under the influence of a force $F : \mathbb{R} \to \mathbb{R}$ is given by **Newton's equation**

$$m\ddot{x}(t) = F(x(t)).$$

   (a) Show that the solutions of this equation are the projections on $\mathbb{R}$ of the flow of the vector field $X \in \mathfrak{X}(T\mathbb{R})$ given by

$$X = v\frac{\partial}{\partial x} + \frac{1}{m}F(x)\frac{\partial}{\partial v},$$

   where (x, v) are the usual local coordinates on $T\mathbb{R} \cong \mathbb{R}^2$.

   (b) Find all force functions $F : \mathbb{R} \to \mathbb{R}$ such that the flow of X commutes with the flow of the vector field

$$Y = x\frac{\partial}{\partial x} + v\frac{\partial}{\partial v}.$$

   What can you say about the solutions of Newton's equations for these forces?

2. The standard Riemannian metric on S^2 can be written as

$$g = d\theta \otimes d\theta + \sin^2\theta d\varphi \otimes d\varphi,$$

where (θ, φ) are the usual spherical coordinates, corresponding to the parameterization $\phi : (0, \pi) \times (0, 2\pi) \to S^2 \subset \mathbb{R}^3$ given by

$$\phi(\theta, \varphi) = (\sin\theta\cos\varphi, \sin\theta\sin\varphi, \cos\theta).$$

   (a) Compute the Christoffel symbols for the Levi-Civita connection in these coordinates.

   (b) Show that the equator is the image of a geodesic. Are the parallels $\theta = \theta_0 \neq \frac{\pi}{2}$ images of geodesics?

   (c) Let $c : [0, 2\pi] \to S^2$ be the curve given in local coordinates by $c(t) = (\theta_0, t)$. Let V be a vector field parallel along c such that $V(0) = \frac{\partial}{\partial\theta}$ (note that $\frac{\partial}{\partial\theta}$ can be extended to the meridian $\varphi = 0$ by continuity). Compute the angle by which V is rotated when it returns to the initial point, that is, compute the angle between $V(0)$ and $V(2\pi)$. (**Remark:** The angle you have computed is exactly the angle by which the oscillation plane of the **Foucault pendulum** rotates during a day in a place at latitude $\frac{\pi}{2} - \theta_0$, as it tries to remain fixed with respect to the distant stars in a rotating Earth).

3. Let $(M, \langle\cdot,\cdot\rangle)$ be a Riemannian manifold with Levi-Civita connection ∇, and let

$$\langle\langle\cdot,\cdot\rangle\rangle = e^{2\rho}\langle\cdot,\cdot\rangle$$

be a metric **conformally related** to $\langle\cdot,\cdot\rangle$, where $\rho \in C^\infty(M)$. Show that the Levi-Civita connection $\widetilde{\nabla}$ of $\langle\langle\cdot,\cdot\rangle\rangle$ is given by

$$\widetilde{\nabla}_X Y = \nabla_X Y + d\rho(X)Y + d\rho(Y)X - \langle X, Y\rangle \operatorname{grad}\rho$$

for all $X, Y \in \mathfrak{X}(M)$, where the gradient is taken with respect to $\langle\cdot,\cdot\rangle$, that is, $\operatorname{grad}\rho$ is the vector field defined by

$$\langle \operatorname{grad}\rho, X\rangle = d\rho(X)$$

for all $X \in \mathfrak{X}(M)$. (**Hint:** Use the Koszul formula).

4. Let $(M, \langle\cdot,\cdot\rangle)$ be a Riemannian manifold. A curve $c : I \subset \mathbb{R} \to M$ is said to be a **reparameterized geodesic** if $c(t) = \gamma(s(t))$ for all $t \in I$, where $\gamma : J \subset \mathbb{R} \to M$ is a geodesic and $s : I \to J$ is a **diffeomorphism** (that is, a smooth bijection with smooth inverse). Show that c is a reparameterized geodesic if and only if it satisfies

$$\frac{D\dot{c}}{dt} = f(t)\,\dot{c}$$

for some differentiable function $f : I \to \mathbb{R}$.