Geometric Mechanics

2022/2023

2$^{\text{nd}}$ Exam - 6 February 2023 - 15:30

Duration: 2 hours

1. The motions of a point particle moving freely on a frictionless platform which is spinning with angular velocity Ω, as seen in the platform's frame, are determined by the Lagrangian $L : T\mathbb{R}^2 \to \mathbb{R}$ given in the usual polar coordinates (r, θ) by

$$L(r, \theta, v^r, v^\theta) = \frac{1}{2}\left((v^r)^2 + r^2(v^\theta)^2\right) + \Omega r^2 v^\theta + \frac{1}{2}\Omega^2 r^2.$$

(3/20) (a) Write the equations of motion, and give an example of a solution.

(2/20) (b) Compute the Legendre transformation and show that L is hyper-regular.

(2/20) (c) Prove that the Hamiltonian $H : T^*\mathbb{R}^2 \to \mathbb{R}$ is completely integrable.

(3/20) (d) The motion of an ice skate on the spinning platform can be modeled by enlarging the configuration space to $\mathbb{R}^2 \times S^1$ and adding the non-holonomic constraint

$$\Sigma = \ker\left(\sin\varphi\, dr - \cos\varphi\, r d\theta\right),$$

where φ is the angular coordinate on S^1, together with a kinetic term of the form $\frac{I}{2}(v^\varphi + v^\theta + \Omega)^2$. Write the equations of motion assuming a perfect reaction force, and give an example of a solution.

(4/20) **2.** Consider an Euler top (that is, a free rigid body with a fixed point) whose principal moments of inertia satisfy $I_1 = I_2 \neq I_3$. Determine the stability of the rotations about each of its principal axes of inertia.

3. The Lorentzian metric describing a (linearized, polarized) **plane gravitational wave** is

$$g = -dt \otimes dt + [1 + \varphi(t - z)]\, dx \otimes dx + [1 - \varphi(t - z)]\, dy \otimes dy + dz \otimes dz,$$

where $\varphi : \mathbb{R} \to \mathbb{R}$ satisfies $|\varphi(u)| \ll 1$ for all $u \in \mathbb{R}$.

(3/20) (a) Show that the curves $c(t) = (t, x_0, y_0, z_0)$, for constant x_0, y_0 and z_0, are time-like geodesics parameterized by proper time (thus corresponding to free-falling observers).

(3/20) (b) Suppose that an observer at $(x, y, z) = (x_0, y_0, z_0)$ sends a light signal towards another observer at $(x, y, z) = (x_0 + \Delta x, y_0, z_0)$, where it is reflected. Show that if $|\Delta x|$ is small enough then the proper time interval Δt measured by the first observer between the emission of the signal and the reception of the reflected signal is approximately

$$\Delta t \simeq [2 + \varphi(t_0 - z_0)]\, \Delta x,$$

where t_0 is the time of emission of the signal.

(**Remark:** The gravitational wave can then be detected by measuring the variation of Δt with the emission time t_0; this is the principle behind modern gravitational wave observatories such as LIGO).