Geometric Mechanics

2022/2023

1^{st} Exam - 24 January 2023 - 13:00
Duration: 2 hours

1. Consider the mechanical system formed by a point particle of $M > 0$ moving along an horizontal line, from which a simple pendulum of length $l > 0$ and mass $m > 0$ has been suspended. This system is described by the Lagrangian $L : T(\mathbb{R} \times S^1) \to \mathbb{R}$ given by

$$L(x, \theta, v^x, v^\theta) = \frac{1}{2}\left((M+m)(v^x)^2 + 2ml\cos\theta v^x v^\theta + ml^2(v^\theta)^2\right) + mgl\cos\theta$$

(where x is the position of point particle and θ is the angle of the pendulum with respect to its stable equilibrium position).

(3/20) (a) Write the equations of motion, and give an example of a non-constant solution.

(2/20) (b) Compute the Legendre transformation and show that L is hyper-regular.

(2/20) (c) Prove that the Hamiltonian $H : T^*(\mathbb{R} \times S^1) \to \mathbb{R}$ is completely integrable.

(3/20) (d) Consider the non-holonomic restriction given by the distribution

$$\Sigma = \ker(dx - l d\theta),$$

Write the equations of motion assuming a perfect reaction force, and give an example of a solution.

(**Remark:** This system models a wheel of radius l, whose center of mass is offset from its center, rolling on the horizontal line).

(4/20) **2.** Let $(M, g, -dU)$ be a conservative mechanical system, and suppose that the vector field $X \in \mathfrak{X}(M)$ satisfies

$$X \cdot U = 0 \quad \text{and} \quad \mathcal{L}_X g = 0.$$

Prove that any motion $c : I \subset \mathbb{R} \to M$ of the system admits the conserved quantity

$$J(t) = g(\dot{c}(t), X_{c(t)}).$$

3. Recall that the restriction of the Schwarzschild metric to a radial line is given by

$$g = -\left(1 - \frac{2m}{r}\right) dt \otimes dt + \left(1 - \frac{2m}{r}\right)^{-1} dr \otimes dr,$$

where $m > 0$ where is the mass of the spherically symmetric object generating the field (we assume $r > 2m$).

(3/20) (a) Show that $(t(r), r)$ is a reparameterized null geodesic for this metric if and only if

$$\frac{dt}{dr} = \pm \left(1 - \frac{2m}{r}\right)^{-1}.$$

(3/20) (b) Suppose that a stationary observer at $r = r_1$ sends a light signal towards $r = r_0 < r_1$, where it is reflected. Show that if $r_0 \gg 2m$ then the proper time interval $\Delta\tau$ measured by the stationary observer at $r = r_1$ between the emission of the signal and the reception of the reflected signal is approximately

$$\Delta\tau \simeq 2\left[\Delta r - m\frac{\Delta r}{r_1} + 2m \log\left(\frac{r_1}{r_0}\right)\right],$$

where $\Delta r = r_1 - r_0$.

(**Remark:** There are two corrections with respect to the Minkowski result $\Delta\tau = 2\Delta r$: a negative term, corresponding to the gravitational time dilation, and a positive term, due to the geometry of the null geodesics, called the **Shapiro delay**; both have been measured in Solar System experiments by reflecting radar pulses off Venus).