Geometric Mechanics

Homework 9

Due on November 25

1. Recall that the motion of a particle of mass $m > 0$ in a central field is given by the completely integrable Hamiltonian $H : T^*\mathbb{R}^2 \to \mathbb{R}$ written in polar coordinates as

$$H\left(r, \theta, p_r, p_\theta\right) = \frac{{p_r}^2}{2m} + \frac{{p_\theta}^2}{2mr^2} + u(r).$$

   (a) Show that there exist circular orbits of radius $r_0 > 0$ whenever $u'(r_0) > 0$.

   (b) Check that the set of points where dH and dp_θ are **not** independent is the union of these circular orbits.

   (c) Check that the projection of the invariant set $L_{(E,l)} = H^{-1}(E) \cap {p_\theta}^{-1}(l)$ onto $\mathbb{R}^2$ is determined by the condition

$$u(r) + \frac{l^2}{2mr^2} \leq E.$$

   (d) Argue that if $u'(r_0) > 0$ and

$$u''(r_0) + \frac{3u'(r_0)}{r_0} > 0$$

   the the circular orbit of radius r_0 is stable.

   (**Hint:** Use the fact that if $f_l(r) = u(r) + \frac{l^2}{2mr^2}$ has a nondegenerate minimum at $r = r_0$, that is, $f_l'(r_0) = 0$ and $f_l''(r_0) > 0$, then the same is true for $f_{\tilde{l}}(r)$ at a nearby point $r = \tilde{r}_0$, as long as $\tilde{l}$ is sufficently close to l).

2. Consider the sequence formed by the first digit of the decimal expansion of each of the integers 2^n for $n \in \mathbb{N}_0$:

$$1, 2, 4, 8, 1, 3, 6, 1, 2, 5, 1, 2, 4, 8, 1, 3, 6, 1, 2, 5, \ldots$$

   The purpose of this exercise is to answer the following question: is there a 7 in this sequence?

   (a) Show that if $\nu \in \mathbb{R} \setminus \mathbb{Q}$ then

$$\lim_{n \to +\infty} \frac{1}{n+1} \sum_{k=0}^{n} e^{2\pi i \nu k} = 0.$$

   (b) Prove the following discrete version of the Birkhoff Ergodic Theorem: if a smooth function $f : \mathbb{R} \to \mathbb{R}$ is periodic with period 1 and $\nu \in \mathbb{R} \setminus \mathbb{Q}$ then for all $x \in \mathbb{R}$

$$\lim_{n \to +\infty} \frac{1}{n+1} \sum_{k=0}^{n} f(x + \nu k) = \int_0^1 f(x) dx.$$

   (c) Show that $\log 2$ is an irrational multiple of $\log 10$.

   (d) Is there a 7 in the sequence above?