Geometric Mechanics

Homework 8

Due on November 18

1. Let M be an n-dimensional differentiable manifold, $\omega \in \Omega^2(T^*M)$ the canonical symplectic form, $H \in C^\infty(T^*M)$ a smooth function, and $h \in \mathbb{R}$ a regular value of H. Show that if $G \in C^\infty(T^*M)$ satisfies $\{G, H\} = 1$ on some neighborhood of the $(2n-1)$-dimensional submanifold $L_h = H^{-1}(h) \subset T^*M$ then the restriction of $\Omega = \iota(X_G)\omega^n$ to L_h is:

   (a) A volume form (that is, if $\alpha \in L_h$ and $v_1, \ldots, v_{2n-1}$ is a basis for $T_\alpha L_h$ then $\Omega(v_1, \ldots, v_{2n-1}) \neq 0$);

   (b) Preserved by the flow of X_H (that is, $\mathcal{L}_{X_H}\Omega = 0$);

   (c) Independent of the choice of G.

   (**Remark:** This volume form can be used to prove a version of the Poicaré recurrence theorem for the energy level L_h; the local existence of a function G as above is guaranteed by the Darboux Theorem).

2. Let $(M, \langle \cdot, \cdot \rangle)$ be a compact Riemannian manifold. Use the version of the Poicaré recurrence theorem mentioned above, applied to a suitable energy level of the appropriate Hamiltonian $H \in C^\infty(T^*M)$, to show that for each open set $U \subset M$ and each $T > 0$ there exist geodesics $c : \mathbb{R} \to M$ with $\|\dot{c}(t)\| = 1$ such that $c(0) \in U$ and $c(t) \in U$ for some $t \geq T$.

   (**Remark:** Actually, almost all geodesics with initial point in U satisfy this property; can you give an example of a compact Riemannian manifold containing a geodesic that **does not** return to U?)