Geometric Mechanics

2020/2021

2^{nd} Exam - 28 January 2021 - 10:00
Duration: 2 hours

1. If M is the cone of aperture $\alpha \in (0, \frac{\pi}{2})$, parameterized by

$$\phi : (0, +\infty) \times (0, 2\pi) \to \mathbb{R}^3$$
$$\phi(\rho, \varphi) = (\rho \cos\varphi, \rho \sin\varphi, \rho \cot\alpha),$$

then the metric induced on M by the Euclidean metric of $\mathbb{R}^3$ is

$$g = \frac{1}{\sin^2\alpha}\, d\rho \otimes d\rho + \rho^2 d\varphi \otimes d\varphi.$$

Consider a particle of unit mass, moving on this surface under the action of a perfect reaction force and the gravitational potential energy

$$U(x, y, z) = gz.$$

(4/20) (a) Write the equations of motion and check that the "parallels" $P_k := \{\rho = k\}$ are images of motions.

(4/20) (b) Compute the Legendre transformation, show that L is hyper-regular, and prove that the Hamiltonian $H : T^*M \to \mathbb{R}$ is completely integrable.

(3/20) (c) Show that motions with image P_k are stable (that is, motions with sufficiently close initial conditions remain close to P_k).

2. The **Gauss Lemma** guarantees that any 4-dimensional Lorentzian metric g can be locally put in the form

$$g = -dt \otimes dt + \sum_{i,j=1}^{3} h_{ij}(t, x)\, dx^i \otimes dx^j\,,$$

where (h_{ij}) is a symmetric positive definite 3×3 matrix of functions.

(3/20) (a) Show that the curves of constant (x^1, x^2, x^3) are timelike geodesics (for this reason the corresponding free-falling observers are said to form a **locally inertial frame**). Show moreover that each of these geodesics is maximizing, that is, any other timelike curve connecting two of its points has smaller length (proper time).

(3/20) (b) In this locally inertial frame, the **energy** of a future-directed null geodesic c, with tangent vector

$$\dot{c} = \dot{t}\frac{\partial}{\partial t} + \sum_{i=1}^{3} \dot{x}^i \frac{\partial}{\partial x^i},$$

is simply $E = \dot{t} > 0$. Show that in general E is not conserved. In the particular case of a flat FLRW model, where $h_{ij} = a^2(t)\delta_{ij}$, show that E is inversely proportional to the radius of the universe $a(t)$.

(3/20) **3.** Let $(M, \langle \cdot, \cdot \rangle, \mathcal{F} = 0, \Sigma)$ be a mechanical system with vanishing external force and non-holonomic constraint Σ given by $\Sigma_p = \{Z_p\}^\perp$, where $Z \in \mathfrak{X}(M)$ is a vector field without zeros. Show that the motions of this system under a perfect reaction force are geodesics of the connection $\tilde{\nabla}$ defined by

$$\tilde{\nabla}_X Y = \nabla_X Y + \frac{\langle \nabla_X Z, Y \rangle}{\langle Z, Z \rangle} Z$$

for all $X, Y \in \mathfrak{X}(M)$, where ∇ is the Levi-Civita connection of $\langle \cdot, \cdot \rangle$.