Geometric Mechanics

Homework 5

Due on October 26

1. Let $(M, \langle\cdot,\cdot\rangle)$ be a Riemannian manifold with Levi-Civita connection $\widetilde{\nabla}$, and let $(N, \langle\langle\cdot,\cdot\rangle\rangle)$ be a submanifold endowed with the induced metric, with Levi-Civita connection ∇. Let $\widetilde{X}, \widetilde{Y} \in \mathfrak{X}(M)$ be local extensions of $X, Y \in \mathfrak{X}(N)$.

   (a) Show that
   $$\nabla_X Y = \left(\widetilde{\nabla}_{\widetilde{X}} \widetilde{Y}\right)^\top,$$
   where $^\top : TM_{|N} \to TN$ is the orthogonal projection. (**Hint:** Use the Koszul formula).

   (b) The **second fundamental form** of N is the map $B : T_pN \times T_pN \to (T_pN)^\perp$ defined at each point $p \in N$ by
   $$B(X_p, Y_p) := \left(\widetilde{\nabla}_{\widetilde{X}} \widetilde{Y}\right)_p - (\nabla_X Y)_p = \left(\widetilde{\nabla}_{\widetilde{X}} \widetilde{Y}\right)_p^\perp.$$
   Show that B is well defined, symmetric and bilinear.

2. The **spherical pendulum** of length l is the mechanical system defined by a particle of mass $m > 0$ moving in $\mathbb{R}^3$ subject to a constant gravitational acceleration g and the holonomic constraint
   $$N = \{(x, y, z) \in \mathbb{R}^3 \mid x^2 + y^2 + z^2 = l^2\}$$
   (assuming a perfect reaction force).

   (a) Write the equations of motion for the spherical pendulum using spherical coordinates.

   (b) Which parallels of N are (images of) motions of the system?