Geometric Mechanics

Homework 13

Due on January 16

1. In this exercise we study in detail the timelike and null geodesics of the Schwarzschild space-time. By symmetry, it suffices to study the geodesics of the totally geodesic 3-dimensional Lorentzian submanifold $\theta = \frac{\pi}{2}$ (**equatorial plane**), whose metric is

$$g = -\left(1 - \frac{2m}{r}\right) dt \otimes dt + \left(1 - \frac{2m}{r}\right)^{-1} dr \otimes dr + r^2 d\varphi \otimes d\varphi.$$

(a) Show that the equations for a future-directed geodesic (parameterized by proper time if timelike) can be written as

$$\begin{cases} \dot{r}^2 = E^2 - \left(\sigma + \frac{L^2}{r^2}\right)\left(1 - \frac{2m}{r}\right) \\ \left(1 - \frac{2m}{r}\right)\dot{t} = E \\ r^2 \dot{\varphi} = L \end{cases}$$

where $E > 0$ and L are integration constants, $\sigma = 1$ for timelike geodesics and $\sigma = 0$ for null geodesics.

(b) Show that if $L \neq 0$ then $u = \frac{1}{r}$ satisfies

$$\frac{d^2 u}{d\varphi^2} + u = \frac{m\sigma}{L^2} + 3mu^2.$$

(c) For situations where relativistic corrections are small one has $mu \ll 1$, and hence the approximate equation

$$\frac{d^2 u}{d\varphi^2} + u = \frac{m}{L^2}$$

holds for timelike geodesics. Show that the solution to this equation is the conic section given in polar coordinates by

$$u = \frac{m}{L^2}(1 + \varepsilon \cos(\varphi - \varphi_0)),$$

where the integration constants $\varepsilon \geq 0$ and φ_0 are the eccentricity and the argument of the pericenter.

(d) Show that for $\varepsilon \ll 1$ this approximate solution satisfies

$$u^2 \simeq \frac{2m}{L^2}u - \frac{m^2}{L^4}.$$

Argue that timelike geodesics close to circular orbits where relativistic corrections are small yield approximate solutions of the equation

$$\frac{d^2 u}{d\varphi^2} + \left(1 - \frac{6m^2}{L^2}\right)u = \frac{m}{L^2}\left(1 - \frac{3m^2}{L^2}\right),$$

and hence the pericenter advances by approximately
$$\frac{6\pi m}{r}$$
radians per revolution.

(**Remark:** The first success of general relativity was due to this effect, which explained the anomalous **precession of Mercury's perihelion** – 43 arcseconds per century).

(e) Show that if one neglects relativistic corrections then null geodesics satisfy
$$\frac{d^2 u}{d\varphi^2} + u = 0.$$
Show that the solution to this equation is the equation for a straight line in polar coordinates,
$$u = \frac{1}{b}\sin(\varphi - \varphi_0),$$
where the integration constants $b > 0$ and φ_0 are the **impact parameter** (distance of closest approach to the center) and the angle between the line and the x-axis.

(f) Assume that $mu \ll 1$. Let us include relativistic corrections by looking for approximate solutions of the form
$$u = \frac{1}{b}\left(\sin\varphi + \frac{m}{b}v\right)$$
(where we take $\varphi_0 = 0$ for simplicity). Show that v is an approximate solution of the equation
$$\frac{d^2 v}{d\varphi^2} + v = 3\sin^2\varphi,$$
and hence u is approximately given by
$$u = \frac{1}{b}\left(\sin\varphi + \frac{m}{b}\left(\frac{3}{2} + \frac{1}{2}\cos(2\varphi) + \alpha\cos\varphi + \beta\sin\varphi\right)\right),$$
where α and β are integration constants.

(g) Show that for the incoming part of the null geodesic ($\varphi \simeq 0$) one approximately has
$$u = 0 \Leftrightarrow \varphi = -\frac{m}{b}\left(2 + \alpha\right).$$
Similarly, show that for the outgoing part of the null geodesic ($\varphi \simeq \pi$) one approximately has
$$u = 0 \Leftrightarrow \varphi = \pi + \frac{m}{b}\left(2 - \alpha\right).$$
Conclude that φ varies by approximately
$$\Delta\varphi = \pi + \frac{4m}{b}$$
radians along its path, and hence the null geodesic is deflected towards the center by approximately
$$\frac{4m}{b}$$
radians.

(**Remark:** The measurement of this **deflection of light** by the Sun – 1.75 arcseconds – was the first experimental confirmation of general relativity, and made Einstein a world celebrity overnight).