Geometric Mechanics

Homework 4

Due on October 18

1. A **symmetry** of a rigid body is an isometry $S \in O(3)$ which preserves the mass distribution (i.e. $m(SA) = m(A)$ for any measurable set $A \subset \mathbb{R}^3$). Show that:

   (a) $SI = IS$, where I is the matrix representation of the inertia tensor;

   (b) if S is a reflection with respect to a plane then there exists a principal axis orthogonal to the plane;

   (c) if S is a nontrivial rotation about an axis then that axis is principal;

   (d) if moreover the rotation is not by π then all axes orthogonal to the rotation axis are principal.

2. Recall that the **Lagrange top** is the mechanical system in $SO(3)$ whose kinetic energy in the local coordinates $(\theta, \varphi, \psi, v^\theta, v^\varphi, v^\psi)$ of $TSO(3)$ associated to the Euler angles is

$$K = \frac{I_1}{2}\left(\left(v^\theta\right)^2 + \left(v^\varphi\right)^2 \sin^2\theta\right) + \frac{I_3}{2}\left(v^\psi + v^\varphi\cos\theta\right)^2,$$

   and whose potential energy is

$$U = Mgl\cos\theta.$$

   (a) Write the equations of motion and determine the equilibrium points.

   (b) Show that there exist solutions such that θ, $\dot\varphi$ and $\dot\psi$ are constant, which in the limit $|\dot\varphi| \ll |\dot\psi|$ (**fast top**) satisfy

$$\dot\varphi \simeq \frac{Mgl}{I_3\dot\psi}.$$