Geometric Mechanics

Homework 10

Due on November 29

Recall that the upper half plane $H = \{(x, y) \in \mathbb{R}^2 \mid y > 0\}$ has a Lie group structure, given by the operation

$$(x, y) \cdot (z, w) := (yz + x, yw),$$

and that the hyperbolic plane is obtained by taking the left-invariant metric

$$g := \frac{1}{y^2} (dx \otimes dx + dy \otimes dy) .$$

1. Show that the geodesics of g are determined by the Hamiltonian function $K : T^*H \to \mathbb{R}$ given by
$$K(x, y, p_x, p_y) = \frac{y^2}{2} \left({p_x}^2 + p_y^2\right) .$$

2. Determine the lift to T^*H of the action of H on itself by left translation, and check that it preserves the Hamiltonian K.

3. Show that the functions
$$F(x, y, p_x, p_y) = yp_x \quad \text{and} \quad G(x, y, p_x, p_y) = yp_y$$
are also H-invariant, and use this to obtain the quotient Poisson structure on T^*H/H. Is this a symplectic manifold?

4. Check that the the quotient Poisson structure on T^*H/H is the standard Lie-Poisson structure, that is
$$\{F, G\}(\mu) = \mu([dF, dG]).$$

5. Write an expression for the momentum map for the action of H on T^*H, and use it to obtain a nontrivial first integral I of the geodesic equations. Show that the projection on H of a geodesic for which $K = E$, $p_x = l$ and $I = m$ satisfies the equation
$$l^2x^2 + l^2y^2 - 2lmx + m^2 = 2E.$$
Assuming $l \neq 0$, what are these curves?