Geometric Mechanics

Homework 8

Due on November 12

1. Recall that the Lagrange top is the mechanical system determined by the Lagrangian function $L : TSO(3) \to \mathbb{R}$ given in local coordinates by

$$L = \frac{I_1}{2}\left(\left(v^\theta\right)^2 + \left(v^\varphi\right)^2 \sin^2\theta\right) + \frac{I_3}{2}\left(v^\psi + v^\varphi \cos\theta\right)^2 - Mgl\cos\theta,$$

where (θ, φ, ψ) are the Euler angles, M is the top's mass and l is the distance from the fixed point to the center of mass.

   (a) Compute the Legendre transformation, show that L is hyper-regular and write an expression in local coordinates for the Hamiltonian $H : T^*SO(3) \to \mathbb{R}$.

   (b) Prove that H is completely integrable.

   (c) Show that the solutions with constant θ, $\dot{\varphi}$ and $\dot{\psi}$ found in Homework 4 are stable for $|\dot{\varphi}| \ll |\dot{\psi}|$ if $|\dot{\psi}|$ is large enough.

2. Consider the sequence formed by the first digit of the decimal expansion of each of the integers 2^n for $n \in \mathbb{N}_0$:

$$1, 2, 4, 8, 1, 3, 6, 1, 2, 5, 1, 2, 4, 8, 1, 3, 6, 1, 2, 5, \ldots$$

The purpose of this exercise is to answer the following question: is there a 7 in this sequence?

   (a) Show that if $\nu \in \mathbb{R} \setminus \mathbb{Q}$ then

$$\lim_{n\to+\infty} \frac{1}{n+1} \sum_{k=0}^{n} e^{2\pi i \nu k} = 0.$$

   (b) Prove the following discrete version of the Birkhoff Ergodic Theorem: if a differentiable function $f : \mathbb{R} \to \mathbb{R}$ is periodic with period 1 and $\nu \in \mathbb{R} \setminus \mathbb{Q}$ then for all $x \in \mathbb{R}$

$$\lim_{n\to+\infty} \frac{1}{n+1} \sum_{k=0}^{n} f(x + \nu k) = \int_0^1 f(x)dx.$$

   (c) Show that $\log 2$ is an irrational multiple of $\log 10$.

   (d) Is there a 7 in the sequence above?