Geometric Mechanics

Homework 5

Due on October 22

1. Using the Frobenius theorem, show that an m-dimensional distribution Σ on an n-manifold M is integrable if and only if

$$d\omega^i \wedge \omega^1 \wedge \cdots \wedge \omega^{n-m} = 0 \qquad (i = 1, \ldots, n-m)$$

for all locally defined sets of differential forms $\{\omega^1, \ldots, \omega^{n-m}\}$ whose kernels determine Σ.

2. Recall that our model for an ice skate is given by the non-holonomic constraint Σ defined on $\mathbb{R}^2 \times S^1$ by the kernel of the 1-form $\omega = -\sin\theta dx + \cos\theta dy$.

   (a) Show that the ice skate can access all points in the configuration space: given two points $p, q \in \mathbb{R}^2 \times S^1$ there exists a piecewise smooth curve $c : [0,1] \to \mathbb{R}^2 \times S^1$ compatible with Σ such that $c(0) = p$ and $c(1) = q$. Why does this show that Σ is non-integrable?

   (b) Assuming that the kinetic energy of the skate is

$$K = \frac{M}{2}\left((v^x)^2 + (v^y)^2\right) + \frac{I}{2}\left(v^\theta\right)^2$$

   and that the reaction force is perfect, show that the skate moves with constant speed along straight lines or circles. What is the physical interpretation of the reaction force?

   (c) Determine the motion of the skate moving on an inclined plane, i.e. subject to a potential energy $U = Mg\sin\alpha\, x$.