Geometric Mechanics

Homework 4

Due on October 15

1. (a) Show that if the mass distribution of the reference configuration of a rigid body is symmetric with respect to reflections in a plane containing the origin then there exists a principal axis orthogonal to the reflection plane.

   (b) Determine the principal axes and the corresponding principal moments of inertia of:

      i. a homogeneous rectangular parallelepiped with mass M, sides $2a, 2b, 2c \in \mathbb{R}^+$ and centered at the origin;

      ii. a homogeneous (solid) ellipsoid with mass M, semiaxes $a, b, c \in \mathbb{R}^+$ and centered at the origin. (**Hint:** Use the coordinate change $(x, y, z) = (au, bv, cw)$).

2. Recall that the **Lagrange top** is the mechanical system in $SO(3)$ whose kinetic energy in the local coordinates $(\theta, \varphi, \psi, v^\theta, v^\varphi, v^\psi)$ of $TSO(3)$ associated to the Euler angles is

$$K = \frac{I_1}{2}\left(\left(v^\theta\right)^2 + \left(v^\varphi\right)^2 \sin^2\theta\right) + \frac{I_3}{2}\left(v^\psi + v^\varphi \cos\theta\right)^2,$$

   and whose potential energy is

$$U = Mgl\cos\theta.$$

   (a) Write the equations of motion and determine the equilibrium points.

   (b) Show that there exist solutions such that θ, $\dot{\varphi}$ and $\dot{\psi}$ are constant, which in the limit $|\dot{\varphi}| \ll |\dot{\psi}|$ (**fast top**) satisfy

$$\dot{\varphi} \simeq \frac{Mgl}{I_3\dot{\psi}}.$$