Geometric Mechanics

Homework 11

Due on December 3

1. **Twin paradox**: Twins Alice and Bob part on their 20th anniversary: while Alice stays on the Earth (which is approximately an inertial frame), Bob leaves at 80% of the speed of light towards Planet X, 8 light-years away from the Earth, which he therefore reaches 10 years later (as measured in the Earth's frame). After a short stay, Bob returns to the Earth, again at 80% of the speed of light. Consequently, Alice is 40 years old when they meet again.

   (a) How old is Bob at this meeting?

   (b) How do you explain the asymmetry in the twin's ages? Notice that, from Bob's point of view, he is the one who is stationary, while the the Earth moves away and back again!

   (c) Imagine that each twin has a very powerful telescope. What does each of them **see**? In particular, how much time elapses for each of them as they see their twin experiencing one year?

2. Let $c : \mathbb{R} \to \mathbb{R}^4$ be the motion of a particle in Minkowski spacetime parameterized by the proper time τ.

   (a) Show that
   $$\langle \dot{c}, \dot{c} \rangle = -1$$
   and
   $$\langle \dot{c}, \ddot{c} \rangle = 0.$$
   Conclude that $\ddot{c}$ is the particle's acceleration as measured in the particle's **instantaneous rest frame**, i.e., in the inertial frame (t, x, y, z) for which $\dot{c} = \frac{\partial}{\partial t}$. For this reason, $\ddot{c}$ is called the particle's **proper acceleration**, and $|\ddot{c}|$ is interpreted as the acceleration measured by the particle.

   (b) Compute the particle's motion assuming that it is moving along the x-axis and measures a constant acceleration $|\ddot{c}| = a$.

   (c) Consider a spaceship launched from the Earth towards the center of the Galaxy (at a distance of 30,000 light-years) with $a = g$, where g represents the gravitational acceleration at the surface of the Earth. Using the fact that $g \simeq 1$ year^{-1} in units such that $c = 1$, compute the proper time measured aboard the spaceship for this journey. How long would the journey take as measured from the Earth?