Geometric Mechanics

Homework 10

Due on November 26

1. Recall that the upper half plane $H = \{(x, y) \in \mathbb{R}^2 \mid y > 0\}$ has a Lie group structure, given by the operation
$$(x, y) \cdot (z, w) := (yz + x, yw),$$
and that the hyperbolic plane corresponds to the left-invariant metric
$$g := \frac{1}{y^2} (dx \otimes dx + dy \otimes dy)$$
on H. The geodesics are therefore determined by the Hamiltonian function $K : T^*H \to \mathbb{R}$ given by
$$K(x, y, p_x, p_y) = \frac{y^2}{2} \left({p_x}^2 + p_y^2\right).$$
   (a) Determine the lift to T^*H of the action of H on itself by left translation, and check that it preserves the Hamiltonian K.
   (b) Show that the functions
   $$F(x, y, p_x, p_y) = yp_x \quad \text{and} \quad G(x, y, p_x, p_y) = yp_y$$
   are also H-invariant, and use this to obtain the quotient Poisson structure on T^*H/H. Is this a symplectic manifold?
   (c) Write an expression for the momentum map for the action of H on T^*H, and use it to obtain a nontrivial first integral I of the geodesic equations. Show that the projection on H of a geodesic for which $K = E$, $p_x = l$ and $I = m$ satisfies the equation
   $$l^2x^2 + l^2y^2 - 2lmx + m^2 = 2E.$$
   Assuming $l \neq 0$, what are these curves?

2. Recall that the Euler top is the mechanical system determined by the Lagrangian function $L : TSO(3) \to \mathbb{R}$ given by
$$L = \frac{1}{2}\langle I\Omega, \Omega\rangle,$$
where Ω are the left-invariant coordinates on the fibers resulting from the identifications
$$T_S SO(3) = dL_S(\mathfrak{so}(3)) \cong \mathfrak{so}(3) \cong \mathbb{R}^3.$$

(a) Show that if we use the Euclidean inner product $\langle \cdot, \cdot \rangle$ to identify $(\mathbb{R}^3)^*$ with $\mathbb{R}^3$ then the Legendre transformation is written

$$P = I\Omega,$$

where P are the corresponding left-invariant coordinates on $T^*SO(3)$.

(b) Write the Hamilton equations on the reduced Poisson manifold $T^*SO(3)/SO(3) \cong \mathbb{R}^3$. What are the symplectic leaves? Give an example of a nonconstant Casimir function.

(c) Compute the momentum map for the lift to $T^*SO(3)$ of the action of $SO(3)$ on itself by left translation.