

Geometric Mechanics

Homework 1

Due on September 24

1. Let $(M, \langle \cdot, \cdot \rangle)$ be a Riemannian manifold with Levi-Civita connection ∇ , and let

$$\langle\langle \cdot, \cdot \rangle\rangle = e^{2\rho} \langle \cdot, \cdot \rangle$$

be a metric conformally related to $\langle \cdot, \cdot \rangle$, where $\rho \in C^\infty(M)$. Show that the Levi-Civita connection $\tilde{\nabla}$ of $\langle\langle \cdot, \cdot \rangle\rangle$ is given by

$$\tilde{\nabla}_X Y = \nabla_X Y + d\rho(X)Y + d\rho(Y)X - \langle X, Y \rangle \text{grad } \rho$$

for all $X, Y \in \mathfrak{X}(M)$, where the gradient is taken with respect to $\langle \cdot, \cdot \rangle$, that is,

$$\langle \text{grad } \rho, X \rangle = d\rho(X)$$

for all $X \in \mathfrak{X}(M)$.

(Hint: Use the Koszul formula).

2. Let $(M, \langle \cdot, \cdot \rangle)$ be a Riemannian manifold. A curve $c : I \subset \mathbb{R} \rightarrow M$ is said to be a **reparameterized geodesic** if $c(t) = \gamma(s(t))$ for all $t \in I$, where $\gamma : J \subset \mathbb{R} \rightarrow M$ is a geodesic and $s : I \rightarrow J$ is a diffeomorphism. Show that c is a reparameterized geodesic if and only if it satisfies

$$\frac{D\dot{c}}{dt} = f(t) \dot{c}$$

for some differentiable function $f : I \rightarrow \mathbb{R}$.