Geometric Mechanics

2015/2016

1^{st} Exam - 21 January 2016 - 10:00

1. Let $M = \mathbb{R}^2 \setminus \{0\}$ and consider the Lagrangian function $L : TM \to \mathbb{R}$ given in polar coordinates by

$$L(r, \theta, v^r, v^\theta) = \frac{1}{2}\left((v^r)^2 + r^2(v^\theta)^2\right) - \frac{1}{2}\left(r^2 + \frac{1}{r^2}\right),$$

which models, in appropriate units, an elastic ring of variable radius r rotating around its center in the plane that contains it (θ is the rotation angle).

(2/20) (a) Write the Euler-Lagrange equations.

(2/20) (b) Determine all solution with constant r, and in particular all equilibrium points.

(2/20) (c) Show that L is hyper-regular, and compute the Hamiltonian function $H : T^*M \to \mathbb{R}$.

(2/20) (d) Show that H is completely integrable.

(2/20) (e) Prove that all the solutions with constant r found in (b) are stable.

(2/20) (f) Consider now the system formed by two such elastic rings rolling without slipping against each other, so that $r_1\dot{\theta}_1 + r_2\dot{\theta}_2 = 0$. Show that this is a true non-holonomic constraint, and write the equations of motion assuming a perfect reaction force.

(2/20) **2.** Let G be a Lie group with Lie algebra $\mathfrak{g}$. Recall that the quotient of the (Poisson) action of G on T^*G by left translations can be identified with $\mathfrak{g}^*$ with the Poisson bracket

$$\{F, H\}(\mu) = \mu([dF, dH]),$$

where $dF, dH \in T^*_\mu\mathfrak{g}^* \simeq \mathfrak{g}$. Assume that there exists a nondegenerate symmetric bilinear form $\langle\cdot,\cdot\rangle$ on $\mathfrak{g}$ such that

$$\langle[\xi, \eta], \zeta\rangle + \langle\eta, [\xi, \zeta]\rangle = 0$$

for all $\xi, \eta, \zeta \in \mathfrak{g}$, as is the case if the group is semisimple (Killing form). Show that in this case the function $C : \mathfrak{g}^* \to \mathbb{R}$ defined by

$$C(\mu) = \langle\mu^\sharp, \mu^\sharp\rangle,$$

where $\mu^\sharp \in \mathfrak{g}$ is the vector associated to $\mu \in \mathfrak{g}^*$ by $\langle\cdot,\cdot\rangle$, is a Casimir function.

3. Recall that the Schwarzschild metric is given by

$$g = -\left(1 - \frac{2m}{r}\right) dt \otimes dt + \left(1 - \frac{2m}{r}\right)^{-1} dr \otimes dr + r^2 d\theta \otimes d\theta + r^2 \sin^2\theta d\varphi \otimes d\varphi.$$

In this exercise we will consider equatorial circular curves, that is, curves with $\theta = \frac{\pi}{2}$ and constant r.

(2/20) (a) Show that the conditions for such curves to be timelike geodesics parameterized by proper time are

$$\begin{cases} \ddot{t} = 0 \\ \ddot{\varphi} = 0 \\ r\dot{\varphi}^2 = \frac{m}{r^2}\dot{t}^2 \\ \left(1 - \frac{3m}{r}\right)\dot{t}^2 = 1 \end{cases}.$$

Conclude that massive particles can orbit the central mass in equatorial circular orbits for all $r > 3m$.

(2/20) (b) Compute the period of an equatorial circular orbit as measured by:

(i) An observer at infinity;

(ii) A stationary observer placed at the same altitude as the orbit;

(iii) The observer in orbit.

(2/20) (c) Show that there exists an equatorial circular null geodesic for $r = 3m$. If a stationary observer placed at $r = 3m$ flashes a light, how long does she have to wait before she sees the flash again?