Geometric Mechanics

Homework 4

Due on October 19

1. (a) Show that if the mass distribution of the reference configuration of a rigid body is symmetric with respect to reflections in a plane containing the origin then there exists a principal axis orthogonal to the reflection plane.

   (b) Determine the principal axes and the corresponding principal moments of inertia of:

      i. a homogeneous rectangular parallelepiped with mass M, sides $2a, 2b, 2c \in \mathbb{R}^+$ and centered at the origin;

      ii. a homogeneous (solid) ellipsoid with mass M, semiaxes $a, b, c \in \mathbb{R}^+$ and centered at the origin. (**Hint:** Use the coordinate change $(x, y, z) = (au, bv, cw)$).

2. (*Precession of the equinoxes*) Due to its rotation, the Earth is not a perfect sphere, but an oblate ellipsoid; therefore its moments of inertia are not quite equal, satisfying approximately

$$I_1 = I_2 \neq I_3;$$
$$\frac{I_3 - I_1}{I_1} \simeq \frac{1}{306}.$$

As a consequence, the combined gravitational attraction of the Moon and the Sun disturbs the Earth's rotation motion. This perturbation can be approximately modelled by the potential energy $U : SO(3) \to \mathbb{R}$ given in the Euler angles (θ, φ, ψ) by

$$U = -\frac{\Omega^2}{2}(I_3 - I_1)\cos^2\theta,$$

where $\dfrac{2\pi}{\Omega} \simeq 168$ days.

   (a) Write the equations of motion and determine the equilibrium points.

   (b) Show that there exist solutions such that θ, $\dot{\varphi}$ and $\dot{\psi}$ are constant, which in the limit $|\dot{\varphi}| \ll |\dot{\psi}|$ (as is the case with the Earth) satisfy

$$\dot{\varphi} \simeq -\frac{\Omega^2(I_3 - I_1)\cos\theta}{I_3\dot{\psi}}.$$

   Given that for the Earth $\theta \simeq 23°$, determine the approximate value of the period of $\varphi(t)$.