From Topology to General Relativity

Tiago Morgadinho

January 2026

Abstract

In Newtonian mechanics, gravity is described as a force. General Relativity (GR) adopts a different approach, interpreting gravity as the curvature of spacetime. This work will provide a basic mathematical framework necessary to an introduction of GR, explaining concepts like smooth manifold, Affine Connection, Levi-Civita connection,Riemann curvature tensor,Ricci's curvature, Riemannian metric and Lorentz metric. After are provided some applications to GR, like the Einstein field equations, Schwarzschild radius and proper time.

1 Mathematical framework

A collection $\mathcal{T}$ of subsets of M is said to be a topology if it satisfies $\varnothing, M \in \mathcal{T}$ and is closed under arbitrary unions and finite intersections. Given a topology $\mathcal{T}$ on M, the pair $(M, \mathcal{T})$ is a topological space. A topological manifold is a topological space that is Hausdorff, second countable and locally Euclidean. Given $U \subset M$ open and a homeomorphism $\varphi : U \to \mathbb{R}^{\dim M}$, the pair (U, φ) is called a coordinate chart. A smooth atlas is a collection of charts that are compatible, i.e. $\psi \circ \varphi^{-1}$ and $\varphi \circ \psi^{-1}$ are smooth, whose images cover M. Given two atlases $\mathcal{A}_1, \mathcal{A}_2$, we say that $\mathcal{A}_1 \sim \mathcal{A}_2$ if $\mathcal{A}_1 \cup \mathcal{A}_2$ is again a smooth atlas. Given a smooth atlas $\mathcal{A}$ on M we define the associated smooth structure by,

$$\mathcal{D}_{\mathcal{A}} = \{\mathcal{A}_1 : \mathcal{A}_1 \sim \mathcal{A}\}$$

we define a smooth manifold as a triple $(M, \mathcal{T}, \mathcal{D})$ where $(M, \mathcal{T})$ is a topological manifold.

Given a smooth manifold M we define a Riemannian metric g as a smooth $(0, 2)$-tensor field such that g_p defines an inner product, i.e. $g_p(u, u) > 0$ for $u \neq 0$ and $g_p(u, v) = g_p(v, u)$. We can completely characterize a tensor field based on the effect it has on the basis, we write,

$$g = g_{\mu\nu} \, dx^\mu \otimes dx^\nu \tag{1}$$

with $g_{\mu\nu} = g(\partial_\mu, \partial_\nu)$. We define the inverse Riemannian metric as the 2-contravariant tensor field $g^{-1} = g^{\mu\nu}\partial_\mu \otimes \partial_\nu$ such that $g^{\mu\alpha}g_{\alpha\nu} = \delta^\mu_\nu$. A Riemannian manifold is a pair (M, g) where M is a smooth manifold and g is a Riemannian metric on M. If instead of assuming positive definiteness we only assume non-degeneracy, then g is called a pseudo-Riemannian metric.Moreover, (M, g) is called a pseudo-Riemannian manifold. The signature of a metric is the pair (p, q) providing the number of positive and negative eigenvalues of the metric at each point. One of most important examples of a pseudo-Riemannian metric is the Lorentz metric that has signature $(-, +, +, +)$ or $(+, -, -, -)$ depending on the convention used, in this work will be used the first one. A pseudo-Riemannian(Lorentz) Manifold is a pair (M, g) where M is a smooth manifold and g is a pseudo-Riemannian(Lorentz) metric. In GR the spacetime is modeled as being a 4-dimensional Lorentzian manifold. The simplest example of a Lorentz metric is given by flat space time, that is, the Minkowski spacetime,

$$ds^2 = -c^2dt^2 + dx^2 + dy^2 + dz^2, \tag{2}$$

with $ds^2 = g_{\mu\nu}dx^\mu dx^\nu$, a point $p \in M$ is called an event, it represents in GR a specific location in space in a specific moment of time.

The sign of $ds^2 := g(v, v)$ determines causal structure, that is whether a given direction in spacetime $v \in T_pM$, is physically allowed for different types of objects:
- $ds^2 < 0$ (timelike) is allowed for massive objects,
- $ds^2 = 0$ (lightlike) is allowed for massless particles such as light,
- $ds^2 > 0$ (spacelike) is forbidden for any physical object.

An affine connection $\nabla : \mathfrak{X}(M) \times \mathfrak{X}(M) \mapsto \mathfrak{X}(M)$ with $\nabla(X, Y) = \nabla_X Y$ is a map that satisfies,

- (Linearity in first argument)$\nabla_{fX+gY}Z = f\nabla_X Z + g\nabla_Y Z$ with $f, g \in C^\infty(M)$
- (Leibniz rule)$\nabla_X fY = X(f)Y + f\nabla_X Y$
- (Additivity in second argument) $\nabla_X(Y + Z) = \nabla_X Y + \nabla_X Z$

Note that $(fX)_p = f(p)X_p$. Let (M, g) be a pseudo-Riemannian manifold, we define for all $X, Y, Z \in \mathfrak{X}(M)$,

$$(\nabla_X g)(Y, Z) = X(g(Y, Z)) - g(\nabla_X Y, Z) - g(Y, \nabla_X Z)$$

with $g(X, Y)(p) = g_p(X_p, Y_p)$. We say that a connection is compatible with the metric if $(\nabla_X g)(Y, Z) = 0$ for all $X, Y, Z \in \mathfrak{X}(M)$

We define the torsion of two fields as,

$$T(X, Y) = \nabla_X Y - \nabla_Y X - [X, Y] \tag{3}$$

Where $[\cdot, \cdot]$ denotes the Lie Bracket on the space of vector fields. A connection is torsion-free if,

$$T(X, Y) = 0 \quad \forall X, Y \in \mathfrak{X}(M)$$

Theorem 1 *Every pseudo-Riemannian manifold (M, g) has a unique Levi-Civita connection, i.e. a connection that is torsion free and compatible with the metric.*

This connection has a special relation with geodesic curves.

Theorem 2 *Let (M, g) be a pseudo-Riemannian manifold, let ∇ be its associated Levi-Civita Connection and $c : (-\varepsilon, \varepsilon) \to M$ smooth curve. Then the curve c is geodesic if and only if $\nabla_{\dot{c}}\dot{c} = 0$.*

The geodesic curves in spacetime have a fundamental interpretation in physics, as they represent the trajectory of a free falling object influenced only by gravity.

Given a pseudo-Riemannian manifold, (M, g) and its Levi-Civita Connection we define the Riemann curvature tensor,

$$R(X,Y)Z = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z \tag{4}$$

Using Einstein notation we can write,

$$R = R^\rho{}_{\sigma\mu\nu}\, \partial_\rho \otimes dx^\sigma \otimes dx^\mu \otimes dx^\nu \tag{5}$$

we can obtain the Ricci tensor components by contracting the Riemann curvature tensor,

$$R_{\sigma\nu} = R^\rho{}_{\sigma\rho\nu} \tag{6}$$

We define the Ricci's scalar by,

$$R := g^{\mu\nu} R_{\mu\nu}, \tag{7}$$

where $g^{\mu\nu}$ denotes the inverse metric tensor.

2 Application to general relativity

In classical mechanics gravity is formulated as a force. A different approach is taken in general relativity where the gravity is modelled as the curvature of a spacetime manifold, this curvature that is related with the distribution of matter and energy by the Einstein field equations. We define the Einstein tensor by,

$$G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu}. \tag{8}$$

We now present the Einstein field equation,

$$G_{\mu\nu} + \Lambda g_{\mu\nu} = k T_{\mu\nu}. \tag{9}$$

where Λ is the cosmological constant and $k = \frac{8\pi G}{c^4}$ with G being the gravitational constant and c being the speed of light. The EFE left side describes the geometry of the spacetime,in the right side we have a tensor $T_{\mu\nu}$ that represents matter and energy content of spacetime, we have,

$$T = T_{\mu\nu}\, dx^\mu \otimes dx^\nu \tag{10}$$

with $T_{\mu\nu} = T_{\nu\mu}$. T_{00} denotes the energy density T_{01}, T_{02}, T_{03} denote the flux of energy through the x, y and z directions respectively. The other components of the tensor represent pressures in different axis, for instance T_{12} represents how much momentum is being transported from the x direction to the y direction. The coordinates T_{11}, T_{22}, T_{33} represent the pressure associated with the respetive spatial coordinates. An example is a perfect fluid, i.e. a fluid that can be completely characterized based on its rest mass density and its isotropic pressure,

$$T_{\mu\nu} = (\rho + p) u_\mu u_\nu + p g_{\mu\nu} \tag{11}$$

where ρ is the energy density and p is the isotropic pressure, u^μ is the 4 velocity of the fluid. In vacuum spacetime there is no stress i.e. $T_{\mu\nu} = 0$, and therefore,

$$G_{\mu\nu} + \Lambda g_{\mu\nu} = 0 \Rightarrow g^{\mu\nu} G_{\mu\nu} + 4\Lambda = 0$$

since $g^{\mu\nu} G_{\mu\nu} = -R$, therefore we have that in vacuum the scalar Ricci's curvature is given by $R = 4\Lambda$.

In the following analysis we will disregard the cosmological constant term $\Lambda = 0$ and also assume that we are in the vacuum i.e. $T_{\mu\nu} = 0$. Another metric that is well known in GR is the Schwarzschild metric,

$$ds^2 = -\left(1 - \frac{2GM}{rc^2}\right) c^2 dt^2 + \left(1 - \frac{2GM}{rc^2}\right)^{-1} dr^2 + r^2 d\Omega^2$$

where $d\Omega^2 = d\theta^2 + \sin^2\theta d\varphi^2$. This metric is used to describe the distortion of spacetime around an object of Mass M, non rotating sphere. Note that if $M = 0$, it's obtained the Minkowski metric written in Schwarzschild coordinates. The Schwarzschild solution has two singularities $r = 0$ and $r = \frac{2GM}{c^2} := r_s$, the last one is called the Schwarzschild radius. However, the singularity $r = r_s$ comes from a bad choice of coordinates, for a different choice, see Eddington-Finkelstein or Kruskal-Szekeres, we may obtain a definition for $r > 0$. This means that this singularity $r = r_s$ is not a physical singularity, since the curvatures does not blow up, only the metric (See Kruskal-Szekeres coordinates). However the radius r_s is still an important value, since it is the radius for which a object of mass M would create a black hole.

Since we suppose a vacuum solution we must have that the radius of the body in analysis R must be greater than of the point we are analyzing, that is, we must be outside the body $r > R$, therefore in most stars the singularity $r = r_s$ could be ignored, for example the radius of the sun is $R^{sun} \approx 700000$km but has a Schwarzschild radius of only $r_s^{sun} \approx 3$km.

A black hole is a region where the physical radius R is less than or equal to r_s, that is, $R \le r_s$, since below this radius a object collapses leading to a creation of a black hole. The event horizon is at $r = r_s$ and the photon sphere which is a region where light may travel in circular orbits is at $r = \frac{3}{2} r_s$

Another interesting topic of GR is the concept of proper time. A world line of an object is the path that it traces in spacetime. The proper time along a timelike world line is defined as being the time measured by a clock following that line and the proper time between two events on a world line is the change in proper time. The equation below describes the relation between spacetime and proper time when $ds^2 < 0$,

$$d\tau^2 = \frac{ds^2}{c^2} \Rightarrow d\tau = -\frac{ds}{c} \tag{12}$$

We will now provide an example of this concept. Considering the Schwarzschild metric we can obtain the associated proper time,

$$d\tau = \sqrt{-\left(1 - \frac{2GM}{rc^2}\right) dt^2 + \frac{1}{c^2}\left(1 - \frac{2GM}{rc^2}\right)^{-1} dr^2 + \frac{r^2}{c^2} d\Omega^2}$$

Consider two objects, one in the north pole and one in the equator. In order to give an estimate of the proper time we will disregard the relativistic effect of the earth rotation and the fact that it isn't spherical, however we will take earth's rotation movement into account when computing $d\varphi$, we will also disregard the cosmological constant and the density and flux of energy and momentum i.e. $T_{\mu\nu} = 0$.

Of course, earth does not verify these conditions, but in order to provide an example we will consider the previous conditions.

We can consider for the north pole object that $dr = d\Omega = 0$, note that the earth mass is given by $M \approx 5.9742 \cdot 10^{24}$ and that the earth radius at the north pole is $r^N \approx 6.36 \times 10^6$ meters, therefore we obtain $d\tau^N = (1 - 6.9540 \times 10^{-10})dt$. Now for the object in the equator, note that since earth is not spherical the radius at the north pole is different from the one in the equator, which is given by $r^E \approx 6.38 \cdot 10^6$ and now we cannot consider $d\varphi = 0$ since earth is in rotation, we have to obtain the value of $d\theta$ using the angular momentum,

$$\frac{d\varphi}{dt} = \frac{2\pi}{\text{sidereal period}} \tag{13}$$

we obtain $d\varphi = 7.2923 \times 10^{-5} dt$ substituting in the proper time equation we obtain,

$$d\tau^E = (1 - 6.9660 \cdot 10^{-10})dt$$

From a non relativistic standpoint these values should be identical.

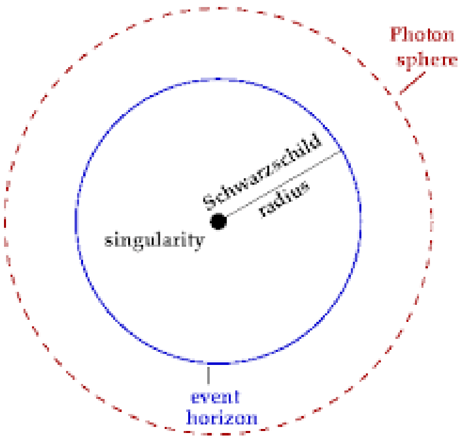

Figure 1: Visualization of the concepts of singularity, Schwarzschild radius an photon sphere.

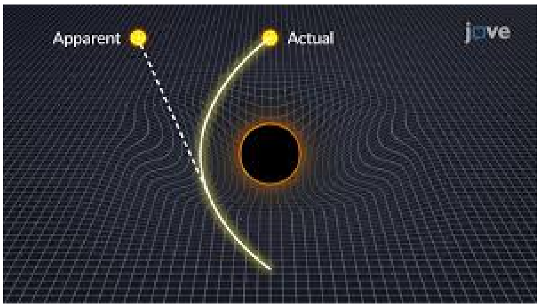

Figure 2: Visualization of the concept of spacetime curvature

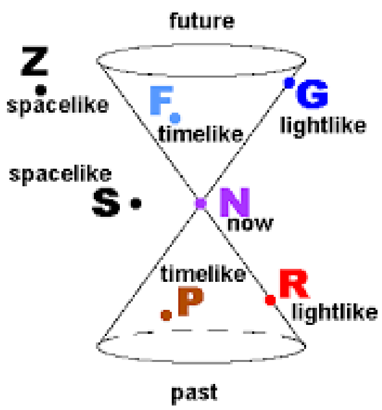

Figure 3: Illustration of the concept of causal structure

References

[1] Schwarzschild Black Holes — Overview, *ScienceDirect Topics*, Elsevier. Available online: `https://www.sciencedirect.com/topics/physics-and-astronomy/schwarzschild-black-holes`.

[2] Natário, José. *General Relativity Without Calculus: A Concise Introduction to the Geometry of Relativity.* Berlin: Springer-Verlag, 2011.

[3] Natário, José. *An Introduction to Mathematical Relativity.* Cham, Switzerland: Springer International Publishing, 2021.

[4] Godinho, Leonor, and José Natário. *An Introduction to Riemannian Geometry: With Applications to Mechanics and Relativity.* Cham, Switzerland: Springer International Publishing, 2014.

[5] Silva, Rui. An introduction to general relativity, Trabalho acadêmico, Instituto Superior Técnico, 2025.