Gauge Theory in QED and some Mathematical Framework

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1 Introduction

Gauge theories are very important in the class of modern quantum field theories, resulting from the demand that a physical system has certain symmetries under transformation of its fields. From this symmetry requirement interaction fields arise automatically. Mathematically, gauge theory makes use of ideas from differential geometry such as connections on bundles, covariant derivatives and curvature tensors as well as Lie groups and Lie algebras.
In this essay we are going to look at gauge theory applied to the example of the Lagrangian in quantum electrodynamics (QED) and at some of the formal mathematical framework.

2 Physical Interpretation

In physics, we describe a Gauge transformation as a transformation of the field Lagrangian interchanging between different so-called gauges. As the physical properties of the system should not change under those transformations, we ask for gauge invariance. This invariance should hold under global transformations as well as local transformations, where the requirements of the local invariance are usually harder to meet as the transformation depends on the point in spacetime.
In the following we are going to use relativistic notation, i.e. $x^\mu = (t, \vec{x})$, $x_\mu = g_{\mu\nu}x^\nu = (t, -\vec{x})$, where the first component denotes the time and the second component the space. The metric g is given by $g_{\mu\nu} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}$ and the derivatives are given by $\partial_\mu = \frac{\partial}{\partial x^\mu} = (\partial_t, \nabla)$, $\partial^\mu = \frac{\partial}{\partial x_\mu} = (\partial_t, -\nabla)$.
Moreover, we use $\hbar = c = 1$.
The space of all gauge transformations on a certain Langrangian forms a Lie group. We differentiate between abelian and non-abelian transformations, depending on whether the Lie group is abelian or non-abelian.
The vector field that appears naturally is called gauge field and if quantized the quanta of the field are called gauge bosons.

Example 1. The theory of QED is an abelian gauge theory with the Lagrangian given by

$$\mathcal{L} = \bar{\psi}D_\mu\psi - m\bar{\psi}\psi - \frac{1}{4}F_{\mu\nu}F^{\mu\nu}.$$

Here it is $D_\mu := \partial_\mu + ieA_\mu$ the gauge covariant derivative, A_μ the electromagnetic vector potential, $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$ the electromagnetic field tensor and $e = -|e|$ the electron charge.
We can arrive at this form of the Lagrangian by taking the Lagrangian for a complex valued Dirac field $\psi(x)$

$$\mathcal{L} = \bar{\psi}(x)(i\gamma^\mu\partial_\mu - m)\psi(x)$$

and asking for invariance under the transformation

$$\psi(x) \to e^{i\alpha(x)}\psi(x),$$

a local phase transformation belonging to the group $U(1)$. The second term of the Lagrangian is already invariant under this transformation. In contrast, we need to manipulate the first term such that it becomes gauge invariant. For this purpose, define D_μ as above. This expression with the vector field can be found by considering infinitesimal gauge transformations. Now the vector field A_μ transforms as $A_\mu \to A_\mu - \frac{1}{e}\partial_\mu\alpha(x)$ and the covariant derivative as follows:

$$D_\mu(x) \to e^{i\alpha(x)}D_\mu(x).$$

Finally, we need to add the kinetic energy term for A_μ which yields $\frac{1}{4}F_{\mu\nu}F^{\mu\nu}$ in the Lagrangian.

3 Mathematical Framework

We are now going to look at the mathematical approach to gauge theory where we define a gauge transformation as a transformation between principle bundles and the gauge fields as connections on those.

Definition 3.1 (Definition)**.** Let G be a Lie group. A **principle bundle** with the structure group G is a triple (P, M, π) where

(i) P and M are smooth manifolds and $\pi : P \to M$ is a surjective submersion.

(ii) G acts on P on the right such that $\pi(p \cdot g) = \pi(p)$ for all $p \in P$, $g \in G$.

(iii) G acts freely and transitively on each fiber $\pi^{-1}(m)$.

(iv) For each $m \in M$ there is a neighbourhood $U \ni m$ and a map ψ_U such that the diagram

$$\begin{tikzcd} \pi^{-1}(U) \arrow[r, "\psi_U"] \arrow[dr, "\pi"'] & U \times G \arrow[d, "pr_1"] \\ & U \end{tikzcd}$$

commutes. Moreover, ψ_U is G-equivariant, where G acts on $U \times G$ by multiplication on the right on the second factor.

Definition 3.2. Let (P, M, π) be a principal bundle. A **global gauge** is a global section of (P, M, π), i.e. a continuous map $s : M \to P$ such that $\pi \circ s = \mathbb{1}_M$.
A **local gauge** is a local section of (P, M, π), i.e. a continuous map $s : U \to P$ with $U \subset M$ open and $\pi \circ s = \iota(U)$, where $\iota : U \to M$ is the inclusion map.

We can now go on to define what a gauge transformation is.

Definition 3.3. A **global gauge transformation** is a bundle automorphism of P, i.e. a diffeomorphism $f : P \to P$ which preserves the fibres of P, $\pi \circ f = \pi$, and is G-equivariant, $f(p \cdot g) = f(p) \cdot g$ for all $p \in P$, $g \in G$. The set of all gauge transformation is denoted by $\mathcal{G}(P)$ or $\mathrm{Aut}(P)$.
A **local gauge transformation** is a bundle automorphism on the principal G-bundle $\pi : P_U \to U$ where $U \subset M$ is an open set.

A gauge transformation therefore transforms between different actions of G on the fibres without changing the physics of the system.
Going back to our example, our base manifold is spacetime, the Lie group $U(1)$ serves as the structure group of the gauge transformation acting on the fibers and the gauge field A_μ becomes a so called connection 1-form, which we are not going to define here, as it would go beyond the scope of this essay. Instead, we can have a quick look at the Standard Model of Particle Physics.
In the Standard Model the underlying structure Lie group is given by $SU(3) \times SU(2) \times U(1)$. It is a non-abelian gauge theory where QED emerges as the abelian part. The Standard Model is one of many theories that are a generalization of QED theory. This generalization consists mainly of constructing gauge invariant Lagrangians under different symmetry groups. In the case of the Standard Model the symmetry groups are those above.
Considering only the group $SU(2)$, as an example for non-abelian gauge theory, yields the Yang-Mills Lagrangian

$$\mathcal{L} = \bar{\psi}(i\gamma^\mu D_\mu)\psi - \frac{1}{4}(F^i_{\mu\nu})^2 - m\bar{\psi}\psi$$

where we transform $\psi(x) \to V(x)\psi(x)$ with $V(x) = \exp i\alpha^i(x)\frac{\sigma^i}{2}$, σ^i the Pauli matrices, $D_\mu = \partial_\mu - igA^i_\mu \frac{\sigma^i}{2}$ and $F^i_{\mu\nu} = \partial_\mu A^i_\nu - \partial_\nu A^i_\mu + g\varepsilon^{ijk} A^j_\mu A^k_\nu$.
As we can see, Gauge theories provide a generalized approach to symmetries and are interesting to study mathematically as well as physically.

4 References

References

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