$SU(2)$ representations and applications in particle physics

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January, 2026

1 Introduction

In this essay we will lay out how algebraic structures emerging from representations of $SU(2)$ allow for the prediction of rates of transition between different quantum states. Mainly, we will explore weights, raising/lowering operators and CG coefficients (Clebsch-Gordan coefficients).

2 $SU(2)$ and representations

We start by understanding the Lie group $SU(2)$, consisting of 2×2 unitary matrices with determinant one, and its associated Lie algebra.. The associated Lie Algebra of this group is given by the commutation relations between its three generators (T_1, T_2 and T_3) :

$$[T_a, T_b] = i\epsilon_{abc}T_c.,$$

from where it comes the familiar commutation relations for angular momentum in QM (using $\hbar = 1$) [1].

We can define two operators from these generators, called raising/lowering operators (the naming will later become clear):

$$T_\pm = T_1 \pm iT_2.$$

These operators, together with T_3 now follow the relations:

$$[T_3, T_\pm] = \pm T_\pm, \quad [T_+, T_-] = 2T_3.$$

Next, we try to find Casimir operator(s), so fulfilling the condition $[C, T_a] = 0$ (with $a = \{1, 2, 3\}$). Taking $J^2 = \sum_a T_a^2$ we can find this fulfills $[J^2, T_a] = 0$, so we can simultaneously diagonalize T_3 and J^2, so they have common eigenvalues that we can use to label the states of $SU(2)$:

$$J^2|j, m\rangle = j(j+1)|j, m\rangle, \quad T_3|j, m\rangle = m|j, m\rangle.$$

Now we notice the action of T_3 on states of the form $T_\pm|j, m\rangle$:

$$T_3(T_\pm|j, m\rangle) = (T_\pm T_3 + [T_3, T_\pm])|j, m\rangle = (m \pm 1)|j, m\rangle,$$

so $T_\pm|j, m\rangle$ must be proportional to $|j, m \pm 1\rangle$, so we parametrize these states as $T_\pm|j, m\rangle = d^\pm_{j,m}|j, m \pm 1\rangle$, and we see that $T_\pm$ raise/lower the states as their name suggests. These parameters can be calculated and yield the following result:

$$d^\pm_{j,m} = \sqrt{(j \mp m)(j \pm m + 1)}.$$

3 Tensor products of $SU(2)$ and CG coefficients

It is of interest to analyse what happens if we take the combination of various $SU(2)$'s, namely two ($R = R_1 \otimes R_2$, R_1 and R_2 are representations of $SU(2)$). The states then take the form $|j_1, m_1\rangle \otimes |j_2, m_2\rangle$, which can be decomposed into the sum of irreducible representations of the total J:

$$|j_1, m_1\rangle \otimes |j_2, m_2\rangle = \sum_{J,M} |J, M\rangle C^{J,M}_{j_1,m_1,j_2,m_2},$$

where $C^{J,M}_{j_1,m_1,j_2,m_2} = \langle J, M|j_1, m_1, j_2, m_2\rangle$ are the CG coefficients [1]. However, we still do not know what these coefficients are. To do so, we resort to the methods of highest weights (this can be analogously done starting with the lowest weight as well). We start with the state of highest weight (so $T_+|J, J\rangle = 0$, we cannot raise the state further), and apply the lowering operator $T^R_-|J, J\rangle = d^-_C|J, J-1\rangle$ ($T^A_\pm$ denote the raising/lowering operators of representation A). Then we use the fact that we can write $T^R_\pm$ with the raising/lowering operators of each $SU(2)$ as $T^R_\pm = T^{R_1}_\pm \otimes Id_{R_2} + Id_{R1} \otimes T^{R2}_\pm$. Since the highest weight state cannot be raised further, then, by using the last relation, we find that it must also be the cross product of highest weight states of both $SU(2)$'s ($|J, J\rangle = |j_1, j_1\rangle \otimes |j_2, j_2\rangle$). We then apply the lowering operator of R in terms of the operators of R_1 and R_2 to the tensor product notation of the highest weight:

$$\begin{aligned}(T^{R_1}_\pm \otimes Id_{R_2} + Id_{R1} \otimes T^{R2}_\pm)(|j_1, j_1\rangle \otimes |j_2, j_2\rangle) = (T^{R_1}_\pm + T^{R2}_\pm)|j_1, j_1, j_2, j_2\rangle = \\ = d^-_{j_1,m_1}|j_1, j_1 - 1, j_2, j_2\rangle + d^-_{j_2,m_2}|j_1, j_1, j_2, j_2 - 1\rangle.\end{aligned}$$

Finally, we can extract $C^{J,J-1}_{j_1,j_1-1,j_2,j_2}$ and $C^{J,J-1}_{j_1,j_1,j_2,j_2-1}$ by aplying to the above equation $\langle j_1, j_1 - 1, j_2, j_2|$ and $\langle j_1, j_1, j_2, j_2 - 1|$ respectivelly. Following this procedure we can obtain all CG coefficients.

4 Application of CG coefficients in physics

We will now briefly describe how we can use the CG coefficient to obtain the realtion of decay width of different processes. In QM and particle physics, the probability of transition between two states is determined by the square of the matrix element of the interaction Hamiltonian (denoted as H). So to a process $A \to B + C$:

$$\Gamma_{A\to B} \propto |M_{A\to B+C}|^2 = |\langle A|\hat{H}|B + C\rangle|^2.$$

When H is spin or isospin invariant (in other words, it respects the symmetry of $SU(2)$), the overlap of the initial and final states is the CG coefficients. As an example, we can take the decays $\Delta^{++} \to p + \pi^+$ and $\Delta^+ \to n + \pi^+$ and compare their rate:

$$\frac{\Gamma_{\Delta^{++}\to p+\pi^+}}{\Gamma_{\Delta^+\to n+\pi^+}} = \left|\frac{\langle \frac{3}{2}, \frac{3}{2}|\frac{1}{2}, \frac{1}{2}; \frac{1}{1}, \frac{1}{1}\rangle}{\langle \frac{3}{2}, \frac{1}{2}|\frac{1}{2}, -\frac{1}{2}; \frac{1}{1}, \frac{1}{1}\rangle}\right|^2 = \frac{1}{(\frac{1}{\sqrt{3}})^2} = 3.$$

Acknowledgments

This work was mainly drafted from the contents of the CU's Group Theory I and Group Theory II in order to illustrate in a simple way how results and logic used in QM arise from the structure of the SU(2) group. As such, i would like to credit professor Andreas Trautner for the examples of aplications of group theory and the explanation of its connection with QM and particle physics.

References

[1] Howard Georgi. *Lie algebras in particle physics: from isospin to unified theories*. Taylor Francis, 2000.