A (brief) introduction to General Relativity and an (even briefer) introduction to Quantum Field Theory in Curved Spacetime

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1 Motivation

General relativity (GR) is a geometric theory of space, time, and gravity, in which gravitational phenomena are interpreted as manifestations of spacetime curvature. Rather than acting as a force in the Newtonian sense, gravity is encoded in the geometric structure of spacetime itself.

In physics, the concept of a field is introduced to describe how physical systems influence one another across distance. To develop a causal theory of gravity, GR replaces the Newtonian gravitational force with a gravitational field encoded in the spacetime metric, which determines distances, time intervals, and the motion of matter [6].

Quantum Field Theory in Curved Spacetime (QFTCS) is a theory where matter is treated according to the principles of QFT, while gravity is treated classically in accord with GR. Due to its classical treatment of the metric, QFTCS cannot be a fundamental theory of nature. However, it is expected to provide an accurate description of quantum phenomena in a regime where the effects of curved spacetime may be significant, such as the description of quantum phenomena occurring in the early universe and near and inside of black holes [5].

The goal of this essay is to introduce some geometric tools needed to understand the basis of General Relativity and how it is related to Quantum Field Theory in Curved Spacetime, as well as some applications of QFTCS.

2 Mathematical Concepts behind GR

GR is deeply rooted in differential geometry, the branch of mathematics that deals with smooth manifolds.

In general relativity, spacetime is modelled as a four-dimensional differentiable manifold $\mathcal{M}$, which is a Hausdorff topological space locally homeomorphic to $\mathbb{R}^4$. The corresponding maps ϕ_i: $\mathcal{O}_i \to \mathbb{R}^4$ defined on overlapping open sets must be compatible and smooth (C^∞) [3, 6]. The maps ϕ_i are called charts, and the collection of charts is called an atlas. Physical laws in general relativity are formulated in a generally covariant manner, meaning they are invariant under smooth coordinate transformations (diffeomorphisms).

We will need to consider smooth functions over a manifold $\mathcal{M}$, so it is also useful to define a tangent vector X_p as an object that differentiates functions at a point $p \in \mathcal{M}$. Specifically, $X_p : C^\infty(\mathcal{M}) \to \mathbb{R}$ satisfies linearity, the Leibniz rule, and $X_p(f) = 0$ when f is constant [6]. Tangent vectors tell us how things change in a given direction. In this case, the set of all tangent vectors at a point p forms a four-dimensional vector space, the tangent space $T_p(\mathcal{M})$.

Although the manifold itself is not a vector space, the tangent space provides a local linear approximation to spacetime.

The central geometric object of general relativity is the spacetime metric. While, in a topological sense, a metric d on a set M is a map $d : M \times M \to \mathbb{R}_0^+$, that satisfies symmetry and triangle inequality, and is positive definite, in GR, the metric is defined as a tensor that captures the geometric and causal structure of spacetime, meaning it assigns an inner product to each tangent space and determines lengths, angles and causal relations [6].

The metric is an inner product on each vector space $T_p(\mathcal{M})$. By definition, a metric g is a symmetric non-degenerate (0,2) tensor field that, with a choice of coordinates, can be written as

$$g = g_{\mu\nu}(x)dx^\mu \otimes dx^\nu = ds^2$$

with ds^2 the line element.

Since the metric $g_{\mu\nu}$ is a symmetric matrix, at each point, one can always locally choose a basis e_μ of each $T_p(\mathcal{M})$ so that this matrix is diagonal. The non-degeneracy condition ensures that none of the diagonal elements vanishes and the number of negative entries gives the signature of the metric [6, 4].

In GR, we are interested in manifolds whose metric has one negative diagonal entry. These manifolds are called Lorentzian and one of the most famous examples of this type of metric is the Minkowski spacetime, whose metric in Cartesian coordinates on $\mathbb{R}^4$, has the form

$$\eta = -dx^0 \otimes dx^0 + dx^1 \otimes dx^1 + dx^2 \otimes dx^2 + dx^3 \otimes dx^3$$

in the $(-, +, +, +)$ signature convention.

At any point p on a general Lorentzian manifold, it is always possible to find an orthonormal basis e_μ of $T_p(\mathcal{M})$ such that, locally, the metric looks like the Minkowski metric [6].

When the metric is locally equivalent to the Minkowski metric, we can import some ideas from special relativity. At any point p, a vector $X_p \in T_p(\mathcal{M})$ is said to be timelike if $g(X_p, X_p) < 0$, null if $g(X_p, X_p) = 0$, and spacelike if $g(X_p, X_p) > 0$.

To define the curvature, it is first necessary to define the notion of connection [6]. A connection (or covariant derivative) on $\mathcal{M}$ is a map $\nabla : \mathfrak{X}(\mathcal{M}) \times \mathfrak{X}(\mathcal{M}) \to \mathfrak{X}(\mathcal{M})$, usually written as $\nabla(X, Y) = \nabla_X Y$, which assigns to a pair of vector fields another vector field, satisfying the following properties for all vector fields X, Y, Z and smooth functions f, g: $\nabla_X(Y + Z) = \nabla_X Y + \nabla_X Z$; $\nabla_{(fX+gY)}Z = f\nabla_X Z + g\nabla_Y Z$ and $\nabla_X(fY) = f\nabla_X Y + (\nabla_X f)Y$

Since the covariant derivative allows one to compare vectors at nearby points, it endows the manifold with additional geometric structure.

There are two tensor fields associated with the connection: the torsion T, a rank $(1, 2)$ tensor, and the curvature R, a rank $(1, 3)$ tensor [6]. The curvature measures the noncommutativity of covariant derivatives and is defined as

$$R(X, Y) = \nabla_X \nabla_Y - \nabla_Y \nabla_X - \nabla_{[X,Y]}$$

In general relativity, the curvature encodes the gravitational field and describes how spacetime deviates from flat Minkowski spacetime.

When one has both a connection and a metric, we can apply the fundamental theorem of Riemannian geometry, which is also valid for Lorentzian geometries [6]. This theorem states that there exists a unique, torsion-free connection that is compatible with a metric g, such that $\nabla_X g = 0$ for all vector fields X.

3 GR and QFTCS

General relativity provides a classical description of spacetime geometry, where the metric is a dynamical field determined by Einstein's equations. On the other hand, QFT describes matter in terms of operator-valued fields defined on a fixed spacetime background [1]. Quantum Field Theory in Curved Spacetime arises in an intermediate regime where these two frameworks overlap: spacetime is treated as a classical Lorentzian manifold, while matter fields propagating on this background are quantised. In this approach, the geometric structures introduced in GR, like the metric, connection, and curvature, are fixed background data that influence the dynamics of quantum fields.

Since Quantum Field Theory in Curved Spacetime is a generalisation of Quantum Field Theory in flat space, some features of QFTCS can be directly inferred from the flat space theory. The differences between curved and flat spacetime arise from the fact that a curved spacetime refers to a Lorentzian manifold whose curvature tensor does not vanish, meaning it cannot be globally identified with Minkowski spacetime [1].

For example, in Minkowski space, the vacuum is unambiguously determined by Poincaré invariance. The Poincaré group is a ten-parameter group of translations, rotations and boosts that leave the Lorentzian manifold invariant, by preserving the line element ds^2. This allows for a preferred notion of time translation and a unique definition of the vacuum state [2]. This also guarantees that the speed of light is the same in every inertial frame.

However, the concept of vacuum is ambiguous in curved space. In a general curved spacetime, there is no Poincaré invariance, meaning there are no global time-translation symmetries, which leads to inequivalent definitions of the vacuum, although spacetime is locally Minkowski. Mathematically, these differences arise from the Bogoliubov transformations, which mix creation and annihilation operators [1]. Because of this, the notion of a particle is observer-dependent and, thus, ambiguous.

3.1 Applications

Even though QFTCS is not a fundamental theory of nature, it has important applications in cosmology, since the fluctuations that caused the creation of galaxies are thought to have a quantum origin, and in black hole thermodynamics [1, 5].

For the second law of thermodynamics to remain valid in the presence of black holes, the creation of particles by black holes is necessary. This process of radiation and evaporation of black holes is an important facet in the fundamental search for a microscopic explanation of the entropy of black holes [1].

The Hawking temperature emerges from the comparison of different notions of time used by observers near the black hole horizon and observers at infinity. This can be done using simulations and toy models adapted to the Schwarzschild geometry, a common black hole background [1].

When a quantum field is quantised on a black hole background, such as the Schwarzschild spacetime, the presence of the event horizon leads to inequivalent definitions of positive-frequency modes [1]. By introducing coordinates such as the tortoise coordinate and the Eddington–Finkelstein or Kruskal–Szekeres null coordinates, one can define field modes that are regular across the horizon. Solving the wave equation in these coordinate systems yields complete sets of incoming and outgoing modes associated with different vacuum states, the Boulware vacuum and the Kruskal vacuum, respectively [1]. The Boulware vacuum is defined using natural modes adapted to an observer sitting very far from the black hole and contains no particles as seen by such observers, but it is singular at the event horizon. In contrast, the Kruskal vacuum is the appropriate one for an observer sitting next to the black hole horizon, which is regular on the future horizon, leading to a nonvanishing particle flux at infinity. The relation between these different mode decompositions is described by Bogoliubov transformations, which mix positive and negative frequency modes and result in a thermal particle spectrum [1]. This allows one to associate a temperature with the black hole, the Hawking temperature.

References

[1] Pieter Bomans. *Quantum Field Theory in Curved Space-Time*. 2023. URL: https://courses.maths.ox.ac.uk/pluginfile.php/97540/mod_resource/content/5/Lecture_Notes_QFT-CS.pdf.

[2] Gernot Eichmann. *QCD and hadron physics*. Appendix B. 2014. URL: http://cftp.ist.utl.pt/~gernot.eichmann/2014-hadron-physics/hadron-app-2.pdf.

[3] Thomas Haworth. *An Introduction to Differential Geometry and General Relativity - A collection of notes for PHYM411*. 2010. URL: http://www.astro.ex.ac.uk/people/haworth/IDGGR.pdf.

[4] et al Hobson M. P. *General Relativity - An Introduction for Physicists*. Cambridge, 2006.

[5] Lionel Mason. *Quantum Field Theory in Curved Space-Time*. 2019. URL: https://www.math.tecnico.ulisboa.pt/~jnatar/nonarxivpapers/Mason_QFT.pdf.

[6] David Tong. *General Relativity- University of Cambridge Part III Mathematical Tripos*. 2019. URL: https://www.damtp.cam.ac.uk/user/tong/gr/gr.pdf.