Hyperbolic Bloch Theory

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1 Motivation

The purpose of this report is to extend, in a simple form, Bloch theory from Euclidean lattice theory to hyperbolic space, utilizing tools from the course of AGMEP. For this purpose, the model is built from group theory, such that certain parallelisms can be drawn when extending to hyperbolic space. The geometry of the hyperbolic space is defined via the Poincaré Disk model, with the group of discrete isometries defined afterwards. Lastly the hyperbolic Bloch theory is shown.

The applications of this model are not restricted to mere theoretical considerations. Firstly, in the past years, circuit quantum electrodynamics have been demonstrated to be modelled by hyperbolic Bloch theory. These types of circuits are of particular interest to study superconductivity as well as quantum computing [1]. Secondly, a few special lattices have been demonstrated to possess constant flat energy bands, a phenomena not yet understood, and the properties of hyperbolic energy bands, which also possess flat bands, provide a useful framework to assess this phenomena [2].

2 Bloch Theory for Euclidean Lattices

Standard Bloch theory for tight-binding periodic euclidean lattices is the cornerstone for most of modern condensed matter physics [3]. The theory consists of finding a fundamental domain which contains a finite number of lattice points, with linearly independent n-dimensional vectors called lattice vectors. In this theory, the set of translations between each lattice site forms a group. Furthermore, it is expected that the wavefunction satisfies the Schrödinger equation and that the translation operator commutes with the Hamiltonian operator.

$$[\hat{T}_{\vec{x}}, \hat{H}] = 0 \tag{1}$$

The wavefunction of the system will be generated by the lattice vector. For 2 dimensions, we can write this vector as follows:

$$\vec{x}(n_1, n_2) = n_1\vec{a} + n_2\vec{b} \quad \text{with} \quad n_1, n_2 \in \mathbb{Z} \tag{2}$$

We then impose that the translation operator shifts the wavefunction onto the next lattice site as follows:

$$\begin{aligned}
\hat{T}_{\vec{a}}\psi(n_1, n_2) &= \psi(n_1 + 1, n_2) \\
\hat{T}_{\vec{b}}\psi(n_1, n_2) &= \psi(n_1, n_2 + 1) \\
\hat{T}_{\vec{a}}\hat{T}_{\vec{b}}\psi(n_1, n_2) &= \psi(n_1 + 1, n_2 + 1) = \\
= \hat{T}_{\vec{b}}\psi(n_1 + 1, n_2) &= \hat{T}_{\vec{b}}\hat{T}_{\vec{a}}\psi(n_1, n_2)
\end{aligned} \tag{3}$$

Where $\hat{T}_{\vec{a}}$ is the translation along the $\vec{a}$ of the lattice vector and $\hat{T}_{\vec{b}}$ is the translation along the $\vec{b}$. The set of $\hat{T}_{\vec{x}}$ forms a group, where translation in each direction is isomorphic to the $(\mathbb{Z}, +)$ group. The full translation group can thus be written as $\mathbb{Z} \times \mathbb{Z}$.

The next common step is to define periodic boundary conditions, such that the translation on site N returns to site 1. There is a periodicity modulo N. Then the set of translations along one direction can be expressed as the quotient group $\mathbb{Z}/N\mathbb{Z} \cong \mathbb{Z}_N$ for a square lattice of size $N \times N$, the translation group with periodic boundary conditions is written as $\mathbb{Z}_N \times \mathbb{Z}_N$. It is important to note that, by equation 3, this group is Abelian.

Since the group is Abelian, by the corollary of Schur's Lemma, the irreducible representations will have dimension 1. Furthermore, due to the modulo N, a group element in one direction to the power of N will be equal to 1, such that the elements will be roots of 1. Then the group elements can be represented as a phase shift factor along each dimension:

$$\begin{aligned}
\left(e^{i\theta}\right)^N &= 1 \\
e^{ik_a a}\psi(n_1, n_2) &= \psi(n_1 + 1, n_2) = \hat{T}_{\vec{a}}\psi(n_1, n_2) \\
e^{ik_b b}\psi(n_1, n_2) &= \psi(n_1, n_2 + 1) = \hat{T}_{\vec{b}}\psi(n_1, n_2) \\
\hat{T}_{\vec{x}} &= \exp\left[i\vec{k} \cdot (n_1\vec{a} + n_2\vec{b})\right]
\end{aligned} \tag{4}$$

We have thus deduced Bloch's theorem from group theory. The considerations taken in this section will be utilized to extend this model onto non-Euclidean surfaces.

3 The Poincaré Disk Model

To extend from the Euclidean geometry to hyperbolic geometry, the Poincaré Disk Model is utilized [4]. This model can be defined from a (2,1) Minkowski metric, where a hyperboloid of two sheets is defined as follows:

$$y^2 = -y_0^2 + y_1^2 + y_2^2 = -R^2, \quad \text{with} \quad y_0, y_1, y_2, R \in \mathbb{R} \tag{5}$$

The focus of this report will lie on the upper sheet with $R = 1$. This surface has a constant negative curvature, which will be proven shortly.

To arrive at the disk model, the following reparametrization of the Minkowski metric takes place.

$$\begin{aligned}
y_0 &= R\cosh\tau \\
y_1 &= R\sinh\tau\cos\phi \\
y_2 &= R\sinh\tau\sin\phi
\end{aligned} \tag{6}$$

Where (τ, ϕ) are polar coordinates of the hyperbolic surface, and the reparametrization obeys the initial

Minkowski (2,1) sphere equation. Thus, setting $R = 1$, the metric is defined in equation 7.

$$\mathrm{d}s^2 = -\mathrm{d}y_0^2 + \mathrm{d}y_1^2 + \mathrm{d}y_2^2 \rightarrow (\mathrm{d}\tau^2 + \sinh^2\tau \,\mathrm{d}\phi^2) \quad (7)$$

Definition 3.1: For a line element of the following form $\mathrm{d}s^2 = \mathrm{d}\tau^2 + f(\tau)^2\mathrm{d}\phi^2$, the Gaussian curvature K at a the point (τ, ϕ) is given by

$$K = -\left(\frac{1}{f}\right)\left(\frac{\partial^2 f}{\partial \tau^2}\right) \quad (8)$$

.

Prop 3.1: The curvature of the hyperbolic surface is constant and negative.

Proof: It is trivial to verify that using the metric defined in equation 7 and definition 3.1, $K = -1$ Q.E.D.

The representation of the hyperbolic surface and its negative curvature can be assessed from figure 1, with the lower circle being the plane projection of a line on the hyperbolic surface. Utilizing the coordinates of this projection, we can again change variables, extending into the complex plane by setting

$$\tau = 2\mathrm{arctanh}\left(\sqrt{x_1^2 + x_2^2}\right)$$
$$\phi = \arg(\mathrm{x}_1 + \mathrm{ix}_2) = \arctan(x_2/x_1) \quad (9)$$

This allows the possibility of computing the geodesic distance in terms of $z = r\exp(i\varphi) = x_1 + ix_2$ as well as compute simple transformations on this variable which will be related to the group actions on the hyperbolic surface. The complex variables written above allow the writing of the more known form of the Poincaré Disk Model: $\mathbb{H}^2 = \{z \in \mathbb{C} : |z| < 1\}$ equipped with the following metric:

$$\mathrm{d}s^2 = 4\frac{\mathrm{d}x_1^2 + \mathrm{d}x_2^2}{(1 - |z|^2)^2} \quad (10)$$

4 The Fuchsian Group

When developing the Euclidean Bloch theory, the set of translation operators acting on the wavefunction of the lattice was seen to be an Abelian group, where we could define the elements as a phase shift. This means the order through which the electron moved in the lattice structure did not matter. However the same will not be true in hyperbolic geometry. There is no translational invariance between points. Instead, in the hyperbolic plane, there are transformations which leave the y_0 sign unchanged, defining the group of isometries of the hyperboloid sphere. To construct the hyperbolic Bloch theory, we make use of the group that these transformations form. Because the Poincaré Disk Model is set on the Minkowski metric it inherits this group of transformations that preserve the y_0 sign.

Definition 4.1: The orthocronous Lorentz group denoted as $SO^+(2,1)$ is a Lie group that preserves the form $(y_0, y_1, y_2) \rightarrow -y_0^2 + y_1^2 + y_2^2$. All orthocronous isometries can be generated by two types - Euclidean rotations by an angle and boosts of rapidity along the direction (y_1, y_2).

However the orthocronous Lorentz group is infinite, and does not produce a lattice structure of discrete points. We require a group Γ such that $\Gamma \subset SO^+(2,1)$ is discrete such that repeated action of this group tiles $\mathbb{H}^2$ with a reference unit cell along the surface. Such a group is known as the Fuchsian group. For the rest of this report, the Poincaré Disk model is restricted to an octagon tessellation $\{8,8\}$ as seen in figure 2. In it, the $\Gamma_{\{8,8\}}$ generators $\gamma_1, \gamma_2, \gamma_3, \gamma_4$ are represented, and replace the translations of Euclidean Bloch theory. The generators obey the following condition [5]:

$$\gamma_1\gamma_2^{-1}\gamma_3\gamma_4^{-1}\gamma_1^{-1}\gamma_2\gamma_3^{-1}\gamma_4 = 1 \quad (11)$$

This group, unlike the one for Euclidean Bloch theory is non-Abelian. Defining $\chi(\gamma_i)$ as the irreducible representation for each generator, we can write the generalized Bloch condition as equation 12, [6].

$$\psi(\gamma(z)) = \chi(\gamma)\psi(z) \quad (12)$$

5 Hyperbolic Bloch Theory

With the Poincaré Disk and the generalized Bloch condition which is acted on by the representation serving as translations in the hyperbolic surface, all that is left is to define Schrödinger's equation to this wavefunction. For this we define the Hamiltonian operator in euclidean quantum mechanics.

Definition 5.1: The Hamiltonian operator in quantum mechanics is defined as $\hat{H} \propto -\nabla^2$ for a free particle.

Restructuring the operator for hyperbolic space means to restructure the Laplacian operator. This is achieved by defining the so called Laplace-Beltrami operator on the Poincaré Disk.

Definition 5.2: The Laplace-Beltrami operator is defined as follows:

$$\Delta = \frac{1}{4}\left(1 - |z|^2\right)^2\left(\frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2}\right) \quad (13)$$

As such, the Bloch wavefunction for hyperbolic space can be defined by equation 14 from, [7].

$$-\Delta\psi = E\psi$$
$$\psi(\gamma(z)) = \chi(\gamma)\psi(z) \quad (14)$$

References

[1] A. Blais, A.L. Grimsmo, S.M. Girvin, A. Wallraff, Rev. Mod. Phys. **93**, 025005 (2021)

[2] A.J. Kollár, M. Fitzpatrick, A.A. Houck, Nature **571**, 45 (2019)

[3] A.J. Kollár, M. Fitzpatrick, P. Sarnak, A.A. Houck, Communications in Mathematical Physics **376**, 1909 (2020)

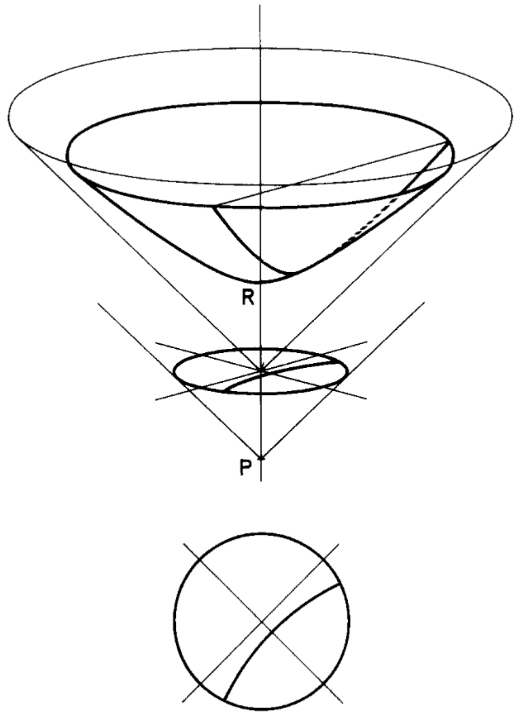

Figure 1: Projection of the hyperboloid onto the unit circle. From [4]

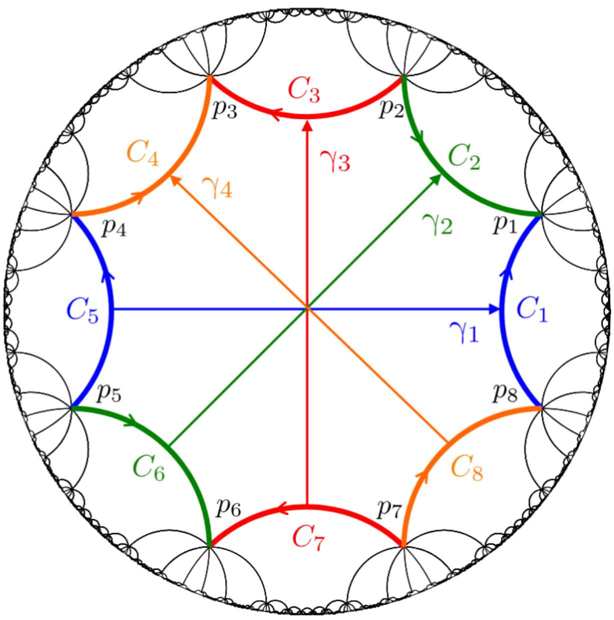

Figure 2: Tessellation of the Poincaré Disk model into 8,8. The group generators γ_i, geodesics C_i and lattice points p_i are identified. From [7]

[4] N. Balazs, A. Voros, Physics Reports **143**, 109 (1986)

[5] blackpenredpen, *Introduction to hyperbolic trigonometric functions*, https://www.youtube.com/watch?v=5k699UuAS0g (2018), youTube video, accessed 2026-01-08

[6] J. Maciejko, S. Rayan, Proceedings of the National Academy of Sciences **119** (2022)

[7] J. Maciejko, S. Rayan, Science Advances **7** (2021)