Algebraic and Geometric Methods in Engineering and Physics

Homework 12

Due on December 17

1. Consider the operation

$$(x, y) \cdot (z, w) = (yz + x, yw)$$

defined on the open set

$$H = \{(x, y) \in \mathbb{R}^2 : y > 0\}.$$

   (a) Show that $(H, \cdot)$ is a Lie group.

   (b) Show that the derivative of the left translation map $L_{(x,y)} : H \to H$ at a given point $(z, w) \in H$ is represented by the matrix

$$\left(dL_{(x,y)}\right)_{(z,w)} = \begin{pmatrix} y & 0 \\ 0 & y \end{pmatrix}.$$

   Conclude that the left-invariant vector field $X^V \in \mathfrak{X}(H)$ determined by the vector

$$V = V^1 \frac{\partial}{\partial x} + V^2 \frac{\partial}{\partial y} \in \mathfrak{h} \equiv T_{(0,1)}H$$

   is given by

$$X^V_{(x,y)} = yV^1 \frac{\partial}{\partial x} + yV^2 \frac{\partial}{\partial y}.$$

   (c) Given $V, W \in \mathfrak{h}$, compute $[V, W]$.

   (d) Determine the flow of the vector field X^V.