Algebraic and Geometric Methods in Engineering and Physics

Homework 11

Due on December 11

1. Consider the map $g : T_p\mathbb{R}^2 \times T_p\mathbb{R}^2 \to \mathbb{R}$ given by

$$g\left(v^1\frac{\partial}{\partial x} + v^2\frac{\partial}{\partial y}, w^1\frac{\partial}{\partial x} + w^2\frac{\partial}{\partial y}\right) = v^1w^1 - v^2w^2,$$

and the map $T : T_p\mathbb{R}^2 \times T_p^*\mathbb{R}^2 \to \mathbb{R}$ given by

$$T\left(v^1\frac{\partial}{\partial x} + v^2\frac{\partial}{\partial y}, p_1dx + p_2dy\right) = v^1p_2 + v^2p_1.$$

   (a) Show that g is a covariant 2-tensor on $T_p\mathbb{R}^2$.

   (b) Write g in the basis $\{dx \otimes dx, dx \otimes dy, dy \otimes dx, dy \otimes dy\}$ of the covariant 2-tensors on $T_p\mathbb{R}^2$.

   (c) If $v \in T_p(M)$ then we define the map $\iota(v)g : T_p^*M \to \mathbb{R}$ as $(\iota(v)g)(w) = g(v, w)$. Show that $\iota(v)g \in T_p^*M$, and write its expression in the basis $\{dx, dy\}$.

   (d) Show that T is a mixed tensor of type $(1, 1)$ on $T_p\mathbb{R}^2$.

   (e) Write T in the basis $\{dx \otimes \frac{\partial}{\partial x}, dx \otimes \frac{\partial}{\partial y}, dy \otimes \frac{\partial}{\partial x}, dy \otimes \frac{\partial}{\partial y}\}$ of the mixed tensors of type $(1, 1)$ on $T_p\mathbb{R}^2$.

   (f) If $v \in T_p(M)$ then we define the map $\iota(v)T : T_p^*M \to \mathbb{R}$ as $(\iota(v)T)(\alpha) = T(v, \alpha)$. Show that $\iota(v)T \in T_pM$, and write its expression in the basis $\{\frac{\partial}{\partial x}, \frac{\partial}{\partial y}\}$.

Remark: (c) shows that a covariant 2-tensor can be seen as a linear map $g : T_p\mathbb{R}^2 \to T_p^*\mathbb{R}^2$; (f) shows that a mixed tensor of type $(1, 1)$ can be seen as a linear map $T : T_p\mathbb{R}^2 \to T_p\mathbb{R}^2$.