Algebraic and Geometric Methods in Engineering and Physics

Homework 10

Due on December 3

1. A topological space $(M, \mathcal{T})$ is said to be **path-connected** if for every $x, y \in M$ there exists a continuous function $c : [0,1] \to M$ (a **path**) such that $c(0) = x$ and $c(1) = y$. Show that if $(M, \mathcal{T})$ is path-connected then it is connected.

2. Consider the atlas for the 2-sphere $S^2 = \{(x,y,z) \in \mathbb{R}^3 : x^2 + y^2 + z^2 = 1\}$ given by the local charts (U_N, π_N) and (U_S, π_S), where $U_N = S^2 \setminus \{(0,0,1)\}$, $U_S = S^2 \setminus \{(0,0,-1)\}$ and

$$\pi_N(x,y,z) = \left(\frac{x}{1-z}, \frac{y}{1-z}\right),$$
$$\pi_S(x,y,z) = \left(\frac{x}{1+z}, \frac{y}{1+z}\right),$$
$$\pi_N^{-1}(s,t) = \left(\frac{2s}{1+s^2+t^2}, \frac{2t}{1+s^2+t^2}, \frac{s^2+t^2-1}{1+s^2+t^2}\right),$$
$$\pi_S^{-1}(u,v) = \left(\frac{2u}{1+u^2+v^2}, \frac{2v}{1+u^2+v^2}, \frac{1-u^2-v^2}{1+u^2+v^2}\right).$$

   (a) Show that these local charts are indeed compatible.

   (b) Prove that the function $f : S^2 \to \mathbb{R}$ given by $f(x,y,z) = z$ is smooth.

   (c) Compute $\frac{\partial}{\partial s}_{|p} \cdot f$, where $p = (1,0,0)$.