Algebraic and Geometric Methods in Engineering and Physics

2025/2026

2^{nd} Exam - 4 February 2026 - 15:30

Duration: 2 hours

(9/20) **1.** Recall that the group $A_4 \subset SO(3)$ of the rotations that preserve a regular tetrahedron centered at the origin is composed of 12 elements: the identity, 8 rotations (by $\pm 2\pi/3$) about an axis through a vertex, and 3 rotations (by π) about an axis through the midpoints of two opposite edges.

(a) Is A_4 isomorphic to the dihedral group D_6 of symmetries of a regular hexagon? Why or why not?

(b) Consider the representation ψ of A_4 obtained by multiplying the rotation matrices by vectors in $\mathbb{C}^3$. Show that the character of ψ is given by

$$\chi_\psi(g) = 1 + 2\cos\alpha\,,$$

where α is the angle of the rotation $g \in A_4$.

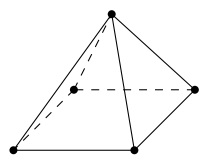
Hint: Recall that a rotation by an angle α about the z-axis is given by the matrix

$$M = \begin{bmatrix} \cos\alpha & -\sin\alpha & 0 \\ \sin\alpha & \cos\alpha & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

(c) Give an example of three elements of A_4 whose conjugacy classes are distinct.

(d) Is ψ irreducible? Why or why not?

(e) How many irreducible finite-dimensional representations does A_4 have? What are their dimensions?



(2/20) **2.** Compute the number of different colorings of the edges of a square pyramid, assuming that each edge can be colored in one of n different colors, and that the pyramid can be freely rotated in space. Check explicitly that your answer is an integer.

(3/20) **3.** Let (M, d) be a metric space and let $x_0 \in M$.

(a) Prove that the function $f : M \to \mathbb{R}$ given by $f(x) = d(x, x_0)$ is continuous.

(b) Show that if $K \subset M$ is a compact set then there exists a point $x_* \in K$ such that $d(x_*, x_0) \leq d(x, x_0)$ for all $x \in K$.

(c) Give an example showing that x_* may not exist if K is not compact.

(6/20) **4.** Consider the matrix Lie group

$$SL(2, \mathbb{R}) = \{A \in M_{2\times 2}(\mathbb{R}) : \det A = 1\}\,.$$

(a) Show that the corresponding Lie algebra is

$$\mathfrak{sl}(2, \mathbb{R}) = \{A \in M_{2\times 2}(\mathbb{R}) : \operatorname{tr} A = 0\}\,.$$

(b) Is $\mathfrak{sl}(2, \mathbb{R})$ simple? Why or why not?

(c) Show that we can define a Lie group representation φ of $SL(2, \mathbb{R})$, and a Lie algebra representation π of $\mathfrak{sl}(2, \mathbb{R})$, both on the real vector space $M_{2\times 2}(\mathbb{R})$, by the formulas

$$\varphi_A(B) = ABA^{-1} \qquad \text{and} \qquad \pi_A(B) = AB - BA\,.$$

Are these representations irreducible?

(d) Given $A \in \mathfrak{sl}(2, \mathbb{R})$, find a vector field $X \in \mathfrak{X}\left(M_{2\times 2}(\mathbb{R})\right)$ such that

$$\varphi_{\exp(tA)}(B) = \psi_t^X(B)$$

for all $B \in M_{2\times 2}(\mathbb{R})$, where ψ_t^X is the flow of X.