Rigid body mechanics using Lie formalism

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INTRODUCTION

In this work, I intend to explain the rigid body dynamics in the framework of Lie groups and Lie algebras. First, I will start by introducing the Special Euclidean group in three dimensions ($SE(3)$) which contains translations and rotations of frames in the Euclidean space and show it is a Lie group. Then, after defining the associated Lie algebra $\mathfrak{se}(3)$, I will define its adjoint representation and compute kinetic quantities such as the velocity, inertia and kinetic energy of a rigid body.

THE SPECIAL EUCLIDEAN GROUP

In order to describe the elements of the $SE(3)$ group, taking into account rotations and translations of a rigid body, one has to define the frames containing the orientation and position of the body in Euclidean space.

The first step is to define a reference frame attached to the rigid body $\{A\}$ and a lab reference frame $\{B\}$. In order to define the orientation of A with respect to B, we construct the rotation matrix $R = [\mathbf{x}_{ab}, \mathbf{y}_{ab}, \mathbf{z}_{ab}] \in SO(3)$ with the usual properties of the group:

$$RR^t = R^tR = I, \ \det(R) = 1 \tag{1}$$

To describe the position of the body, one simply defines the position vector $p \in \mathbb{R}^3$ of the origin of A with respect to the origin of B. Therefore, it is possible to represent an element of $SE(3)$ as a pair (R, p) with the operation given by $(\cdot, +)$, where $\cdot$ is the matrix product. The identity of the group is $e = (I, 0)$

Then, one can consider the homogeneous representation of $SE(3)$ as:

$$T_{(R,p)} = \begin{bmatrix} R & p \\ 0 & 1 \end{bmatrix}, \tag{2}$$

4×4 matrix with

$$T^{-1}_{(R,p)} = \begin{bmatrix} R^t & -R^tp \\ 0 & 1 \end{bmatrix}, \tag{3}$$

its inverse.[1]

[1] One must note that the operation between R and p is well defined as the matrix product

It is possible to show that $SE(3)$ is a Lie group because, naturally $SE(3) \cong SO(3) \times \mathbb{R}^3$, with $SO(3)$ and $\mathbb{R}^3$ also Lie groups.

LIE ALGEBRA ASSOCIATED TO $SE(3)$

From the Lie group structure of $SE(3)$ one can deduce that the Lie algebra should be $\mathfrak{se}(3) \cong \mathfrak{so}(3) \times \mathbb{R}^3$, since $\mathbb{R}^3$ is its own Lie algebra[2]. $\mathfrak{so}(3)$ is the tangent space at the identity of $SO(3)$ and the elements follow the properties $\hat{w} = -\hat{w}^t, \ \mathrm{tr}(\hat{w}) = 0$. These properties can be shown by recalling the relation between the algebra and the group, through the exponential map. Then, the elements are given by

$$\hat{w} = \begin{bmatrix} 0 & -w_3 & w_2 \\ w_3 & 0 & -w_1 \\ -w_2 & w_1 & 0 \end{bmatrix} \tag{4}$$

with w_i the independent components. Similarly to the convention made when describing the elements of $SE(3)$, the elements of $\mathfrak{se}(3)$ can be written as the pair $(\hat{w}, v), \ \hat{w} \in \mathfrak{so}(3), \ v \in \mathbb{R}^3$. The Lie bracket of $\mathfrak{se}(3)$ is, then, given by:

$$\left[(\hat{w}, v), (\hat{\theta}, u)\right] = \left(\left[\hat{w}, \hat{\theta}\right], 0\right), \ \forall \hat{w}, \hat{\theta} \in \mathfrak{so}(3) \\ \forall v, u \in \mathbb{R}^3 \tag{5}$$

with $\left[\hat{w}, \hat{\theta}\right] = -\left[\hat{w}, \hat{\theta}\right]^t$ the commutator between $\hat{w}$ and $\hat{\theta}$. The translational part of the Lie bracket is 0 because $\mathbb{R}^3$ is an abelian group, so $[v, u] = (v + u) - (u + v) = 0, \ \forall v, u \in \mathbb{R}^3$. Since the dimension of $\mathfrak{so}(3)$ is $\dim(\mathfrak{so}(3)) = \frac{3^2-3}{2} = 2$, the dimension of $\mathfrak{se}(3)$ is $\dim(\mathfrak{se}(3)) = 2 \times 3 = 6$, considering that the dimension of $\mathbb{R}^3$ is 3.

Recalling the choice of T as the homogeneous representation of the group, one can represent the elements of the algebra as:

$$\xi = \begin{bmatrix} \hat{w} & v \\ 0 & 0 \end{bmatrix}, \tag{6}$$

a 4×4 matrix. The zero in the last entry ensures that the trace of the matrix continues to be zero.

[2] The tangent space in the identity of $\mathbb{R}^3$ is itself

The adjoint representation

In order to describe the adjoint representation of $\mathfrak{se}(3)$, one can start by considering the adjoint action of the element $T \in SE(3)$ on $\xi \in \mathfrak{se}(3)$ as $\mathrm{Ad}_T(\xi) = T\xi T^{-1}$. Thus, for the representations defined above:

$$\mathrm{Ad}_T(\xi) = \begin{bmatrix} R\hat{w}R^t & Rv - R\hat{w}R^t p \\ 0 & 0 \end{bmatrix} \tag{7}$$

This shows that rotations transform as $\hat{w} \to R\hat{w}R^t$ and translations as $v \to Rv - R\hat{w}R^t p$.

Choosing (w, v) as the natural basis for $\mathfrak{se}(3)$, where $w = (w_1, w_2, w_3)$ is the vector whose components are given by the skew-symmetric matrix, $\hat{w}$, independent entries, we can compute[3]:

$$\begin{aligned}(R\hat{w}R^t)\mathbf{x} = R(w \times (R^t\mathbf{x})) = (Rw) \times (RR^t\mathbf{x}) = \\ = (Rw) \times \mathbf{x},\end{aligned} \tag{8}$$

$$\begin{aligned}-R\hat{w}R^t p = -R(\hat{w} \times (R^t p)) = -(Rw) \times (RR^t p) \\ = p \times (Rw) = \hat{p}Rw,\end{aligned} \tag{9}$$

where $\hat{p}$ is the skew-symmetric matrix associated with the translation vector p[4]:

$$\hat{p} = \begin{bmatrix} 0 & -p_3 & p_2 \\ p_3 & 0 & -p_1 \\ -p_2 & p_1 & 0 \end{bmatrix} \tag{10}$$

Using this formalism Ad_T is written in this basis as:

$$\mathrm{Ad}_T = \begin{bmatrix} R & 0 \\ \hat{p}R & R \end{bmatrix}, \tag{11}$$

called the adjoint representation.

GENERALIZED QUANTITIES

Having at disposal the representations of the Euclidean group and algebra, it is possible to define generalized mechanical quantities such as velocity, force, inertia and momentum. One starts by imposing that the $SE(3)$ elements are time dependent, so $R = R(t)$, $p = p(t)$, for instance. Using this formalism, it is possible to compute time derivatives of the representations. Furthermore, it is crucial that these quantities live in the algebra $\mathfrak{se}(3)$ to transform linearly.

I will start by defining the generalized velocity of a rigid body as follows:

$$V = T^{-1}\dot{T} = \begin{bmatrix} R^t\dot{R} & R^t\dot{p} \\ 0 & 0 \end{bmatrix} \tag{12}$$

Notice that, in general, $\dot{T}$ is not well defined as an element of $\mathfrak{se}(3)$, since it can not live in the tangent space at the identity of $SE(3)$. Therefore, the multiplication by T^{-1} ensures that $T \in SE(3)$ is properly mapped to $V \in \mathfrak{se}(3)$ pushing back $\dot{T}$ to the tangent space at the identity.

Alternatively, V can be simply expressed by the 6-dimensional vector

$$V = \begin{pmatrix} \Omega \\ U \end{pmatrix}, \tag{13}$$

where $\hat{\Omega} = R^t\dot{R}$ represents the angular velocity and $U = R^t\dot{p}$ represents the linear velocity of the rigid body. The adjoint transformation defined above (equation 7) acts by changing the reference frames that describe the rigid body:

$$V_b = \mathrm{Ad}_{T_{ba}} V_a, \tag{14}$$

change of coordinates from $\{A\}$ to $\{B\}$.

Kinetic energy and Inertia

In order to define the kinetic energy associated to a rigid body we start by recalling the formula

$$K = \frac{1}{2}\int_{vol} \rho(\mathbf{x}) \|\dot{\mathbf{x}}\|^2 \, d\mathbf{x}, \tag{15}$$

where $\dot{\mathbf{x}} = \Omega \times x + U$ and $\rho(\mathbf{x})$ is the mass density of the rigid body. Expanding, one gets:

$$\begin{aligned}\|\dot{\mathbf{x}}\|^2 = (\Omega \times \mathbf{x} + U)^t(\Omega \times \mathbf{x} + U) = \\ = \mathbf{x}^t\hat{\Omega}^t\hat{\Omega}\mathbf{x} + \mathbf{x}^t\hat{\Omega}^t U + U^t\hat{\Omega}\mathbf{x} + U^t U\end{aligned} \tag{16}$$

Substituting in equation 15 one has that:

$$\begin{aligned}2K =& U^t U \int_{vol} \rho(\mathbf{x}) \, d\mathbf{x} - \Omega^t \left(\int_{vol} \rho(\mathbf{x}) \, \hat{x} \, d\mathbf{x}\right) U - \\ & - U^t \left(\int_{vol} \rho(\mathbf{x}) \, \hat{x} \, d\mathbf{x}\right) \Omega + \Omega^t \left(\int_{vol} \rho(\mathbf{x}) \hat{x}^t \hat{x} \, d\mathbf{x}\right) \Omega\end{aligned} \tag{17}$$

[3] Note that R is not sensitive to the cross product
[4] The ˆ notation must be regarded as the skew-symmetric matrix representation of a given vector

Considering that $\int_{vol} \rho(\mathbf{x})\ d\mathbf{x} = m$ we can define the inertia matrix as:

$$I = \begin{bmatrix} \int_{vol} \rho(\mathbf{x}) \hat{x}^t \hat{x}\ d\mathbf{x} & \int_{vol} \rho(\mathbf{x})\ \hat{x}\ d\mathbf{x} \\ \int_{vol} \rho(\mathbf{x})\ \hat{x}\ d\mathbf{x} & m \end{bmatrix}, \tag{18}$$

a 6×6 matrix. Finally, we can simply write the kinetic energy in the most usual fashion

$$K = \frac{1}{2} V^t I V. \tag{19}$$

It is possible to define other generalized quantities such as force and angular momentum and compute total derivatives in order to get the laws of motion for a rigid body.

CONCLUSIONS

Using Lie formalism is very powerful when it comes to describe mechanical properties of rigid bodies considering its rotations and translations for a given choice of reference frames. After defining the Special Euclidean Group ($SE(3)$) and identifying it as a Lie group with a corresponding Lie algebra, the mechanical quantities that live in it transform over a change of coordinates through the adjoint action. This formalism has several applications such as in robotics, mechanical engineering and physics.

[1] Unknown, Geometric mechanics, chapter 14 (n.d.), accessed: 2025-01-03.

[2] J. Lee, Lie group dynamics and control (2010), accessed: 2025-01-03.

[3] J. E. Marsden and M. West, Discrete mechanics and variational integrators, arXiv preprint math-ph/0402052 (2004), accessed: 2025-01-03.

[4] Unknown, Rigid body motion: Lecture notes 3 (n.d.), accessed: 2025-01-03.