An Introduction to General Relativity

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Introduction

Let $(\mathcal{M}, g)$ be a pseudo-Riemannian manifold, where $\mathcal{M}$ is a connected 4-dimensional differentiable manifold and $g : T_pM \times T_pM \to \mathbb{R}$ is a symmetric, nondegenerate differentiable 2-tensor field with signature $(-,+,+,+)$. Such a manifold, called a **Lorentzian manifold**, provides the mathematical framework for general relativity (GR), describing spacetime geometry.

The goal of this essay is to outline how the curvature of spacetime relates to the distribution of matter and energy through Einstein's field equations (EFE). To this end, I introduce the necessary geometric tools, focusing on curvature and its connection to EFE, and illustrate these ideas through the Schwarzschild solution and gravitational time dilation.

Geometrical Foundations

In Euclidean space, differentiation of vector fields is straightforward because tangent spaces at all points are naturally identified. However, on a general differentiable manifold, tangent spaces at different points are distinct vector spaces, requiring a mechanism to compare vectors across them. This is provided by the **affine connection**:

Definition 1 (Affine Connection)**.** Let M be a differentiable manifold. An **affine connection** is a map $\nabla : \mathfrak{X}(M) \times \mathfrak{X}(M) \to \mathfrak{X}(M)$, where $\mathfrak{X}(M)$ is the set of smooth vector fields on M, satisfying the following properties for all $X, Y, Z \in \mathfrak{X}(M)$ and $f \in C^\infty(M, \mathbb{R})$:

1. $\nabla_{fX+Y} Z = f\nabla_X Z + \nabla_Y Z$,
2. $\nabla_X (Y + Z) = \nabla_X Y + \nabla_X Z$,
3. $\nabla_X (fY) = (X \cdot f)Y + f\nabla_X Y$.

For pseudo-Riemannian manifolds, there exists a unique **Levi-Civita connection**, defined by two properties:

1. **Torsion-free:** $\nabla_X Y - \nabla_Y X - [X, Y] = 0$ for all $X, Y \in \mathfrak{X}(M)$.
2. **Metric compatibility:** $X \cdot g(Y, Z) = g(\nabla_X Y, Z) + g(Y, \nabla_X Z)$ for all $X, Y, Z \in \mathfrak{X}(M)$.

The Levi-Civita connection governs the behavior of geodesics, curves $c : [0, T] \to \mathcal{M}$ satisfying $\nabla_{\dot{c}} \dot{c} = 0$. For such curves, $g(\dot{c}, \dot{c})$ remains constant:

$$\frac{d}{ds} g(\dot{c}(s), \dot{c}(s)) = 2g(\nabla_{\dot{c}} \dot{c}, \dot{c}) = 0. \tag{1}$$

Definition 2 (Curvature)**.** The **curvature tensor** R of a connection ∇ is defined by

$$R(X, Y)Z = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z,$$

where $X, Y, Z \in \mathfrak{X}(M)$. The **Ricci tensor** is obtained by contracting the Riemann tensor:

$$\mathrm{Ric}(X, Y) = \sum_{k=1}^{n} dx^k \big(R(\frac{\partial}{\partial x^k}, X)Y\big),$$

and the **Ricci scalar** is defined as $\mathrm{tr}_g \mathrm{Ric}$. In terms of local coordinates Ricci scalar is given by this formula $g^{\mu\nu} R_{\mu\nu}$ where $g^{\mu\nu}$ are the inverse components of the metric tensor.

While the previous definitions were presented in a coordinate-free manner for their generality and mathematical elegance, the following sections adopt a coordinate-based approach for the sake of simplicity and practical utility in calculations.

Einstein Field Equations

The Einstein field equations (EFE) describe how spacetime curvature relates to matter and energy:

$$G_{\mu\nu} + \Lambda g_{\mu\nu} = \kappa T_{\mu\nu}, \tag{2}$$

where $G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu}$ is the Einstein tensor, where $R_{\mu\nu}$ represents the Ricci tensor in local coordinates and R the Ricci scalar. The energy momentum tensor is represented by $T_{\mu\nu}$, Λ is the cosmological constant, and $\kappa = \frac{8\pi G}{c^4}$ is the Einstein gravitational constant.

Whenever the connection of a Lorentzian Manifold $(\mathcal{M}, g)$ satisfies the equation below. We say that $(\mathcal{M}, g)$ is a vacuum solution for the EFE.

$$R_{\mu\nu} = 0. \tag{3}$$

Since in a vacuum there is no matter therefore the energy momentum tensor must be 0 and by ignoring the cosmological constant (actually Einstein removed the cosmological constant from his equations and was later readded) we can derive that in fact the Ricci tensor must be 0 in the vacuum since

$$R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} = 0 \iff g^{\mu\nu}(R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu}) = 0 \tag{4}$$

$$R - 2R = 0 \tag{5}$$

Concluding that the Ricci scalar must be 0, therefore by replacing the Ricci scalar term in the EFE we conclude that the Ricci tensor must be 0.

Schwarzschild Solution and Physical Interpretation

The Schwarzschild solution, a vacuum solution of the EFE, describes a static, spherically symmetric spacetime. Assuming the metric

$$g = -A(r)^2 dt \otimes dt + B(r)^2 dr \otimes dr + r^2 d\theta \otimes d\theta + r^2 \sin^2\theta d\varphi \otimes d\varphi,$$

and solving (3), we find the Schwarzschild metric:

$$g = -\left(1 - \frac{2m}{r}\right) dt \otimes dt + \left(1 - \frac{2m}{r}\right)^{-1} dr \otimes dr + r^2 d\theta \otimes d\theta + r^2 \sin^2\theta d\varphi \otimes d\varphi, \tag{6}$$

where m is related to the source's mass.

To illustrate gravitational time dilation, consider two observers: one at the Earth's surface ($r_E = r$) and the other in geosynchronous orbit ($r_O = r + h$) for some height $h > 0$. In local Minkowski coordinates their respective metrics reduce to:

$$ds^2 = -d\tau_E^2 \quad ds^2 = -d\tau_O^2$$

Since both are at rest their spatial coordinates are respectively 0 and we only account for time Proper times for processes measured by these observers are given by:

$$d\tau_E = dt\sqrt{1 - \frac{2m}{r}} \quad d\tau_O = dt\sqrt{1 - \frac{2m}{r+h}}.$$

Since $d\tau_E < d\tau_O$, clocks closer to the massive object tick slower. This effect, known as **gravitational time dilation**, is a direct consequence of spacetime curvature.

References

[1] Leonor Godinho and José Natário. *An Introduction to Riemannian Geometry: With Applications to Mechanics and Relativity*. Springer International Publishing, 2014.

[2] Marko Vojinovíc. Schwarzschild solution in general relativity, 2014.