Topology in Ergodic Theory and Dynamical Systems

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1 Introduction

Ergodic theory is a branch of dynamical systems theory which focuses on transformations that preserve a certain measure. This study seeks to answer certain questions about a dynamical system, such as whether points return to their respective initial conditions and the expected time they might take. These questions go back to the work of the Austrian physicist Ludwig Boltzmann responsible for the development of the kinetic theory of gases. In these systems, a large number of particles collide constantly with each other, and due to their sheer number, it would be intractable to apply the laws of classical mechanics to each individual one. Much like its beginnings, this domain of study is tightly linked to its applications in physics, with areas such as statistical mechanics and thermodynamic formalism bridging the gap between pure mathematics and the modeling of physical systems.

2 Recurrence

One of the foundational results of ergodic theory is the Poincaré Recurrence Theorem, which states that points in a system will return arbitrarily close to their initial position, provided the system meets some (quite general) requirements. Poincaré's recurrence theorem is often formulated in measure-theoretic terms and consequently proved with measure-theoretic techniques. However, one can expand this result whilst only requiring simple topological notions. The aim of this essay is to present the topological version of this result and its simple proof, as well as to showcase Birkhoff's recurrence theorem, a purely topological reformulation of Poincaré's recurrence theorem.

3 Poincaré Recurrence Theorem

In this theorem we assume M to be a topological space that admits a countable basis, i.e., a family of open sets $\{U_k : k \in \mathbb{N}\}$ such that any open subset of M

is the union of some U_k's. In addition, M is endowed with it's Borel σ-algebra, that is, the collection of subsets of M closed under complements, countable unions and countable intersections which contains (and is generated by) the open sets. In order to specify what it means for points to "return arbitrarily close to their initial positions", we introduce the notion of recurrence: a point $x \in M$ is recurrent under a transformation $f : M \to M$ if there exists a sequence of natural numbers $n_j \to \infty$ such that $f^{n_j}(x) \to x$. Another important notion mentioned previously is that of transformations that preserve measure or, in other terms, measures which are invariant under some transformation: a measure μ is invariant under f if $\mu(E) = \mu(f^{-1}(E))$ for every measurable set $E \subset M$ (in this notation $f^{-1}(E)$ is the preimage of E by f). We start by stating the original formulation, which we will require for the topological version.

Theorem (Poincaré recurrence theorem - Measurable version)**.** *Let $f : M \to M$ be a measurable transformation and μ be a finite measure on M invariant under f. Let $E \subset M$ be any measurable set with $\mu(E) > 0$. Then for μ-almost every point $x \in E$ there exist infinitely many values of n for which $f^n(x) \in E$.*

Theorem (Poincaré recurrence theorem - Topological version)**.** *Suppose M is a topological space with a countable basis of open sets. Let $f : M \to M$ be a measurable transformation and μ be a finite measure on M invariant under f. Then, μ-almost every $x \in M$ is recurrent for f.*

Proof. For each k let $\tilde{U}_k$ be the set of points in $x \in U_k$ that never return to U_k. According to the measurable version, every $\tilde{U}_k$ has measure zero, which implies that the countable union

$$\tilde{U} = \bigcup_{k \in \mathbb{N}} \tilde{U}_k$$

also has measure zero. Therefore, we just need to check that every point outside $\tilde{U}$ is recurrent. Let $x \in M \setminus \tilde{U}$ and let U be any neighborhood of x. By definition of basis, there exists some U_k such that $x \in U_k$ and $U_k \subset U$. Since x is not in $\tilde{U}$ then x is also not in $\tilde{U}_k$, meaning that x is one of the points in U_k that "returns" to U_k. So there exists $n \geq 1$ such that $f^n(x) \in U_k$ and consequently $f^n(x) \in U$. Since U was an arbitrary neighborhood of x, we can take smaller and smaller neighborhoods of x, which proves the desired result. □

4 Birkhoff Recurrence Theorem

The following is the purely topological version of this result, which is formulated and proved without making use of measure-theoretic concepts.

Theorem (Birkhoff recurrence theorem)**.** *If $f : M \to M$ is a continuous transformation on a compact metric space M then there exists some point $x \in M$ which is recurrent for f.*

Proof. Let I be the family of non-empty closed sets $X \subset M$ that are invariant under f, meaning that $f(X) \subset X$. This family is non-empty, since $M \in I$. In order to prove the desired result we will show that an element $X \in I$ is minimal for the inclusion relation if and only if the orbit of every $x \in X$ is dense in X. If X is closed and invariant then X contains the closure of the orbit of each one of its elements. So, for X to be minimal under inclusion, it must coincide with every one of these closures, so each one of these must be dense in X. Conversely, if X equals the closure of the orbit of each of its elements, then it has no proper subset that is closed and invariant, so it is minimal. Since the orbit of x is dense, there is a sequence $n_j \to \infty$ such that $f^{n_j}(x) \to x$, so every $x \in X$ is recurrent. So, every point in a minimal set is recurrent. The remainder of the proof consists of an application of Zorn's lemma that shows I contains such minimal elements. □

5 Application to Physics

Invariant measures find natural applications in physics. As observed by the French mathematician Joseph Liouville, in classical mechanics every energy-preserving system admits an invariant volume measure in the space of configurations. It was based on this observation that Poincaré, when studying the movement of celestial bodies, came to the realization that bodies display recurrent behavior, that is, they will return arbitrarily close to their initial state in the sense of both position and velocity. Unless, of course, they continue onto infinity. Much like these examples, dynamical systems with invariant measures and Poincaré's recurrence theorem lend themselves to numerous applications in mathematical physics, as do the fields that arose from their study, namely Ergodic Theory, Statistical Mechanics and Thermodynamic Formalism. They would also require more exposition and detail in their treatment so, being such vast domains of study, would be better fitted to the scope of a larger text.

6 Conclusion

Topology's abstract approach to fundamental concepts in mathematics such as sets, transformations and convergence lends itself in a very useful way to the study of dynamical systems. With it we were able to showcase a profound result that, not only has multiple examples in pure mathematics but was also foundational in the development of the study of dynamical systems in mathematical physics.

References

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