Maxwell equation in differential form

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1 Introduction

Reality should not depend on the basis that we choose to describe it. Considering this idea, a natural way to describe reality is with tensors, as these are defined to be independent of any basis. In this context, because Maxwell's equations are invariant under Lorentz transformations, which correspond to rotations in space-time, then electromagnetism aligns with the principles of relativity, being independent on the choice of the reference frame or, for that matter, the choice of basis. In fact, traditionally, Maxwell equations in 3 dimensions are natural and pure mathematical consequences of three very simple physical laws:

1. Principle of Relativity with constant speed of light: invariance under Lorentz transformation
2. Electric field flux lines have thei origin and end at electric charges, and their intensity is proportional to the electric chagre
3. There are no magnetic monopoles

However, the standard vector formulation of Maxwell's equations does not immediately reveal the Lorentz invariance of physical laws. This is because the quantities traditionally used $(\mathbf{E}, \mathbf{B}, \rho, \mathbf{J})$ are not tensors in space-time, and their components do not transform correctly under Lorentz transformations. Instead, we can use alternative, more fundamental quantities, that are combinations of the fields and charges, differential forms in R^{3+1}.

2 Representation of physical quantities in differential forms

2.1 differential forms and exterior derivative

First, consider the two 2-forms and the 3-form in space-time R^{3+1}:

$$\mathbf{\Phi} = \mathbf{E} \wedge \mathrm{d}t + \frac{1}{c}\mathbf{B}, \quad \mathbf{\Psi} = \mathbf{H} \wedge \mathrm{d}t - \frac{1}{c}\mathrm{D}, \quad \mathbf{\Gamma} = \frac{1}{c}(\mathbf{J} \wedge \mathrm{d}t - \rho).$$

where the vector fields of force $\mathbf{E}$ and $\mathbf{B}$ are 1-forms, the vector fields of flux $\mathbf{D}$, $\mathbf{B}$, $\mathbf{J}$ are 2 forms, the field of density ρ is a 3-form (because forces are integrated over lines, fluxes over surfaces and densities over volumes).

We start by applying the exterior derivative on these quantities:

$$\begin{aligned}
\mathrm{d}\mathbf{\Phi} &= \mathrm{d}\left(\mathbf{E} \wedge \mathrm{d}t\right) + \frac{1}{c}\mathrm{d}\mathbf{B} = \mathrm{d}\mathbf{E} \wedge \mathrm{d}t + \frac{1}{c}\mathrm{d}\mathbf{B} \\
&= \left(\mathrm{d}'\mathbf{E} + \dot{\mathbf{E}} \wedge \mathbf{d}t\right) \wedge \mathrm{d}t + \frac{1}{c}\left(\mathrm{d}'\mathbf{B} + \dot{\mathbf{B}} \wedge \mathbf{d}t\right) \\
&= \left(\mathrm{d}'\mathbf{E} + \frac{1}{c}\dot{\mathbf{B}}\right) \wedge \mathbf{d}t
\end{aligned}$$

$$\mathrm{d}\mathbf{\Psi} = \mathrm{d}\left(\mathbf{H} \wedge \mathrm{d}t\right) + \frac{1}{c}\mathrm{d}\mathbf{D} = \mathrm{d}'\mathbf{H} + \frac{1}{c}\dot{\mathbf{D}} \wedge \mathrm{d}t + \frac{1}{c}\mathrm{d}'\mathbf{D}$$

$$\mathrm{d}\Gamma = \frac{1}{c}\mathrm{d}'\mathbf{J} + \dot{\rho} \wedge \mathrm{d}t$$

where d' designates the exterior derivative in space and the dot the time derivative.

The above second and third laws correspond to respectively

$$\mathrm{d}'\mathbf{D} = 4\pi\rho, \quad \mathrm{d}'\mathbf{B} = 0$$

Now, consider a Lorentz transformation along the x direction between the original and the new coordinate systems with tilde.

Since the basis of a covector is contravariant, it has the same transformation of the coordinates of the vectors in the basis, which are, according to the Lorentz transformation:

$$\tilde{x} = \beta(x - vt), \quad \tilde{y} = y, \quad \tilde{z} = z, \quad \tilde{t} = \beta\left(t - \frac{v}{c^2}x\right)$$

$$\mathrm{d}x' = \beta\left(dx - v\,dt\right), \quad \mathrm{d}y' = \mathrm{d}y, \quad \mathrm{d}z' = \mathrm{d}z, \quad \mathrm{d}\tilde{t} = \beta\left(t - \frac{v}{c^2}\mathrm{d}x\right)$$

We can write the 2-forms in the two coordinate systems, and use the transformation relations to substitute the basis with tilde into the original basis:

$$\Phi = (E_1\mathbf{dx} + E_2\mathbf{dy} + E_3\mathbf{dz}) \wedge \mathbf{dt} + \frac{1}{c}(B_{12}\mathbf{dx} \wedge \mathbf{dy} + B_{13}\mathbf{dx} \wedge \mathbf{dz} + B_{23}\mathbf{dy} \wedge \mathbf{dz})$$

$$\Phi = \left(\tilde{E}_1\,\mathbf{d\tilde{x}} + \tilde{E}_2\,\mathbf{d\tilde{y}} + \tilde{E}_3\,\mathbf{d\tilde{z}}\right) \wedge \mathbf{dt} + \frac{1}{c}\left(\tilde{B}_{12}\,\mathbf{d\tilde{x}} \wedge \mathbf{d\tilde{y}} + \tilde{B}_{13}\,\mathbf{d\tilde{x}} \wedge \mathbf{d\tilde{z}} + \tilde{B}_{23}\,\mathbf{d\tilde{y}} \wedge \mathbf{d\tilde{z}}\right)$$

$$\Phi = \beta^2\tilde{E}_1(\mathbf{dx} - v\mathbf{dt}) \wedge (\beta(\mathbf{dt} - \frac{v}{c^2}\mathbf{dx}) + \beta(\tilde{E}_2(\mathbf{dy} + E_3(\mathbf{dz}) \wedge (\beta(\mathbf{dt} - \frac{v}{c^2}\mathbf{dx})$$
$$+\frac{1}{c}[\beta\tilde{B_{12}}(\mathbf{dx} - v\mathbf{dt} \wedge \mathbf{dy} + \beta\tilde{B_{13}}(\mathbf{dx} - v\mathbf{dt}) \wedge \mathbf{dz} + \tilde{B_2}3\mathbf{dy} \wedge \mathbf{dz}]$$

The validity of equation $\mathrm{d}'\mathbf{B} = 0$ is equivalent to

$$\frac{\partial B_{23}}{\partial x} - \frac{\partial B_{13}}{\partial y} + \frac{\partial B_{12}}{\partial z} = 0, \quad \frac{\partial \tilde{B}_{23}}{\partial x} - \frac{\partial \tilde{B}_{13}}{\partial y} + \frac{\partial \tilde{B}_{12}}{\partial z} = 0$$

Using the transformation of components of $\mathbf{B}$ in the first equation we get:

$$(\beta\frac{\partial\tilde{B_{23}}}{\partial\tilde{x}} - \frac{\beta v}{c}\frac{\partial\tilde{B_{23}}}{\partial\tilde{t}}) - (\frac{\beta v}{c}\frac{\partial\tilde{E_3}}{\partial\tilde{y}} + \beta\frac{\partial\tilde{B_{13}}}{\partial\tilde{y}}) + (\frac{\beta v}{c}\frac{\partial\tilde{E_2}}{\partial\tilde{z}} + \beta\frac{\partial\tilde{B_{12}}}{\partial\tilde{z}}) = 0$$

$$\beta(\frac{\partial\tilde{B_{23}}}{\partial\tilde{x}} - \frac{\partial\tilde{B_{13}}}{\partial\tilde{y}} + \frac{\partial\tilde{B_{12}}}{\partial\tilde{z}}) - \frac{\beta v}{c}(\frac{1}{c}\frac{\partial\tilde{B_{23}}}{\partial\tilde{t}} + \frac{\partial\tilde{E_3}}{\partial\tilde{y}} - \frac{\partial\tilde{E_2}}{\partial\tilde{z}}) = 0$$

The 3-form $\mathrm{d}'\mathbf{B}$ equals to the first term in the above equation divided by β and multiplied by $\mathbf{dx} \wedge \mathbf{dy} \wedge \mathbf{dz}$. Therefore, the validity of the equation $\mathrm{d}'\mathbf{B} = 0$ in each coordinate systems is equivalent to the annulation of the second term. Repeating this for Lorentz transformation in y and z directions, we obtain:

$$\begin{aligned}
\frac{1}{c}\frac{\partial\tilde{B_2}3}{\partial\tilde{t}} + \frac{\partial\tilde{E_3}}{\partial\tilde{y}} - \frac{\partial\tilde{E_2}}{\partial\tilde{z}} &= 0,\\
\frac{1}{c}\frac{\partial\tilde{B_{13}}}{\partial\tilde{t}} + \frac{\partial\tilde{E_3}}{\partial\tilde{x}} - \frac{\partial\tilde{E_1}}{\partial\tilde{z}} &= 0,\\
\frac{1}{c}\frac{\partial\tilde{B_1}2}{\partial\tilde{t}} + \frac{\partial\tilde{E_2}}{\partial\tilde{x}} - \frac{\partial\tilde{E_1}}{\partial\tilde{y}} &= 0
\end{aligned}$$

By Multiplying the left side of each equation by resp., $\mathbf{d\tilde{y}} \wedge \mathbf{d\tilde{z}}$, $\mathbf{d\tilde{x}} \wedge \mathbf{d\tilde{y}}$ and $\mathbf{d\tilde{x}} \wedge \mathbf{d\tilde{y}}$, we have the equation $\mathbf{d}'\mathbf{E} + \frac{1}{c}\dot{\mathbf{B}} = 0$, which together with $\mathrm{d}'\mathbf{B} = 0$ is equivalent to $\mathrm{d}\mathbf{\Psi} = 0$. So $\mathrm{d}'\mathbf{B} = 0$ in all inertial coordinate systems related by Lorentz transformation is equivalent to $\mathrm{d}\mathbf{\Psi} = 0$.

Similarly, the validity of the equation $\mathrm{d}'\mathbf{D} = 4\pi\rho$ in the two coordinate systems related by Lorentz transformation is equivalent to the initial equations obtained for $\mathrm{d}'\mathbf{B} = 0$, substituting $\mathbf{B}$ with $\mathbf{D}$ on the right side of the equation 0 by $4\pi\rho$, so we get in the same way that $\mathrm{d}'\mathbf{D} = 4\pi\rho$ in all coordinate systems related by Lorentz transformation is equivalent to $\mathrm{d}\mathbf{\Psi} = 4\pi\Gamma$.

In general we have the following relations:

$$\begin{aligned}
d\Phi = 0 \quad &\Longleftrightarrow \begin{cases} d'\mathbf{E} + \frac{1}{c}\dot{\mathbf{B}} = 0 \\ d'\mathbf{B} = 0 \end{cases} &&\Longleftrightarrow \begin{cases} \nabla \times \mathbf{E} + \frac{1}{c}\dot{\mathbf{B}} = 0 \\ \nabla \cdot \mathbf{B} = 0 \end{cases}\\
d\Psi = 4\pi\Gamma \quad &\Longleftrightarrow \begin{cases} d'\mathbf{H} - \frac{1}{c}\dot{\mathbf{D}} = \frac{4\pi}{c}\mathbf{J} \\ d'\mathbf{D} = 4\pi\rho \end{cases} &&\Longleftrightarrow \begin{cases} \nabla \times \mathbf{H} - \frac{1}{c}\dot{\mathbf{D}} = \frac{4\pi}{c}\mathbf{J} \\ \nabla \cdot \mathbf{D} = 4\pi\rho \end{cases}\\
dT = 0 \quad &\Longleftrightarrow \frac{1}{c}\left(d'\mathbf{J} + \dot{\rho}\right) \wedge dt = 0 &&\Longleftrightarrow \frac{\partial\rho}{\partial t} - \nabla \cdot \mathbf{J} = 0.
\end{aligned}$$

which imply that the Maxwell equation in space-time R^{3+1} are:

$$d\Phi = 0, \quad d\Psi = 4\pi\Gamma$$

3 Observations

Note that the equation of continuity $\mathrm{d}\Gamma = 0$ is automatically satisfied because $\mathrm{d}\Gamma = \mathrm{d}(\mathrm{d}\Psi) = 0$. The equation $\mathrm{d}\Psi = 0$ implies that Ψ is a closed 2-form, and thus it is locally exact, so $\Psi = \mathrm{dA}$, where A is the electromagnetic potential 1-form.

4 Appendix

E: electric field ; **H**: magnetic field ; **D**: electric displacement ; **B**: magnetic induction ; **J**: electric current density ; ρ: charge density ;

Constitutive laws: $\mathbf{D} = \epsilon\mathbf{E}$, $\mathbf{B} = \mu\mathbf{H}$, $\mathbf{J} = \sigma\mathbf{E}$

Lorentz transformations

$$\tilde{x} = \beta(x - vt), \quad \tilde{y} = y, \quad \tilde{z} = z, \quad \tilde{t} = \beta\left(t - \frac{v}{c^2}x\right)$$

$$B_{12} = \frac{\beta v}{c}\tilde{E}_2 + \beta\tilde{B_{12}}, \quad B_{13} = \frac{\beta v}{c}\tilde{E}_3 + \beta\tilde{B_{13}}, \quad B_{23} = \tilde{B_{23}} \tag{1}$$

$$D_{12} = -\frac{\beta v}{c}\tilde{H}_2 + \beta\tilde{D_{12}}, \quad D_{13} = -\frac{\beta v}{c}\tilde{H}_3 + \beta\tilde{D_{13}}, \quad D_{23} = \tilde{D_{23}} \tag{2}$$

The four Maxwell equation in R^{3+1}

$$\nabla \times \mathbf{E} + \frac{\partial\mathbf{B}}{\partial t} = 0, \quad \nabla \cdot \mathbf{D} = \rho, \quad \nabla \cdot \mathbf{D} = \rho, \quad \nabla \cdot \mathbf{B} = 0 \tag{3}$$