On the Symmetries of Many-Body Quantum Mechanics

Gabriel Almeida

In quantum mechanics, identical particles are indistinguishable, imposing strong symmetry constraints on their wave functions. This essay examines these constraints from an algebraic perspective by introducing an S_n representation on the Hilbert space of an n-particle system. We then explore the Hong-Ou-Mandel effect and its extensions, where the bosonic nature of photons causes remarkable changes in the outcome probabilities. Because these probabilities involve computations intractable for classical systems, such experiments have garnered interest as potential demonstrations of quantum advantage.

In classical physics it is possible to keep track of individual particles, even if they look alike. In quantum mechanics, however, identical particles are truly indistinguishable. This is because we can no longer follow each particle trajectory, as this would violate the Heisenberg uncertainty principle. This apparently innocent observation gives rise to striking quantum-mechanical effects.

Consider the wave function[1] of two identical particles $\psi(\mathbf{x_1}, \mathbf{x_2})$ and the particle exchange operator $\hat{P}_{12}\psi(\mathbf{x_1}, \mathbf{x_2}) \equiv \psi(\mathbf{x}_2, \mathbf{x}_1)$. Since the two particles are undistinguishable, the wave function should remain the same (up to a global phase) under particle exchange, that is, $\hat{P}_{12}\psi(\mathbf{x_1}, \mathbf{x_2}) = \eta\psi(\mathbf{x_1}, \mathbf{x_2})$. Since $\hat{P}_{12}^2 = 1$, we have $\eta = \pm 1$. Thus, quantum systems of identical particles are either symmetric or antisymmetric under particle exchange. In the former case the particles are called bosons, while in the latter they are called fermions [1].

Given that nature imposes this symmetry on the wave functions, we can anticipate the existence of deep connections to algebra and group theory. In this essay, we will unravel some of these connections by introducing a S_n representation on the Hilbert space of the particles. We also discuss the Hong-Ou-Mandel effect and its generalizations as a way to demonstrate the practical implications of these symmetry constraints.

S_n REPRESENTATION

In quantum mechanics, the state of a particle is described by a unit vector in a complex Hilbert space $\mathcal{H}$. If there are n particles, the system is described by a unit vector in the tensor product of the Hilbert spaces $\mathcal{H}^{\otimes n}$. We define the representation $S_n \curvearrowright^{\varphi} \mathcal{H}^{\otimes n}$ such that φ_σ is the linear map that acts as follows on the basis states

$$\varphi_\sigma \left|\mathbf{x}_1, \ldots, \mathbf{x}_n\right\rangle = \left|\mathbf{x}_{\sigma^{-1}(1)}, \ldots, \mathbf{x}_{\sigma^{-1}(n)}\right\rangle .$$

Intuitively, φ_σ is the operator that interchanges particles according to permutation σ. This is a representation because

$$\begin{aligned}\varphi_\tau(\varphi_\sigma \left|\mathbf{x}_1, \ldots, \mathbf{x}_n\right\rangle) &= \varphi_\tau \left|\mathbf{x}_{\sigma^{-1}(1)}, \ldots, \mathbf{x}_{\sigma^{-1}(n)}\right\rangle \\ &= \left|\mathbf{x}_{\sigma^{-1}(\tau^{-1}(1))}, \ldots, \mathbf{x}_{\sigma^{-1}(\tau^{-1}(n))}\right\rangle \\ &= \varphi_{\tau\circ\sigma} \left|\mathbf{x}_1, \ldots, \mathbf{x}_n\right\rangle .\end{aligned}$$

Moreover, the representation is unitary since

$$\begin{aligned}&\left\langle \mathbf{x}_1, \ldots, \mathbf{x}_n\right| \varphi_\sigma^\dagger \varphi_\sigma \left|\mathbf{y}_1, \ldots, \mathbf{y}_n\right\rangle \\ &= \left\langle \mathbf{x}_{\sigma^{-1}(1)}, \ldots, \mathbf{x}_{\sigma^{-1}(n)} \middle| \mathbf{y}_{\sigma^{-1}(1)}, \ldots, \mathbf{y}_{\sigma^{-1}(n)}\right\rangle \\ &= \left\langle \mathbf{x}_{\sigma^{-1}(1)} \middle| \mathbf{y}_{\sigma^{-1}(1)}\right\rangle \ldots \left\langle \mathbf{x}_{\sigma^{-1}(n)} \middle| \mathbf{y}_{\sigma^{-1}(n)}\right\rangle \\ &= \left\langle \mathbf{x}_1 \middle| \mathbf{y}_1\right\rangle \ldots \left\langle \mathbf{x}_n \middle| \mathbf{y}_n\right\rangle \\ &= \left\langle \mathbf{x}_1, \ldots, \mathbf{x}_n \middle| \mathbf{y}_1, \ldots, \mathbf{y}_n\right\rangle .\end{aligned}$$

By Maschke's Theorem[2], this representation can be decomposed into irreducible representations. The simplest of these are the trivial representation, φ^S, and the sign representation, φ^A. Clearly, the invariant subspaces where φ acts as φ^S is the subspace of symmetric states, while the invariant subspace where φ acts as φ^A is the subspace of antisymmetric states.

HONG–OU–MANDEL EFFECT

We now discuss the Hong-Ou-Mandel effect [2], which was demonstrated in 1987 by three physicists from the University of Rochester. In this experiment, two photons enter a beam splitter, one in each input mode, and are detected on the output modes (see Fig. 1). When a photon enters a beam splitter, it will either be reflected or transmitted with equal probability. Since the photons do not interact, classical reasoning would predict the following probabilities $P(20) = P(02) = 1/4$ and $P(11) = 1/2$. However, the experiment results showed that $P(11) = 0$ and $P(20) = P(02) = 1/2$. This counterintuitive outcome arises from quantum interference, where the symmetric wave function of bosons (the photons) causes the $P(11)$ term to cancel out.

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[1] Here, $\mathbf{x}$ is a vector with all the quantum numbers such as position, spin, or polarization needed to describe each particle

[2] To apply the theorem, we assume a finite Hilbert space $\mathcal{H}$, ensuring the decomposition of φ into irreducible representations.

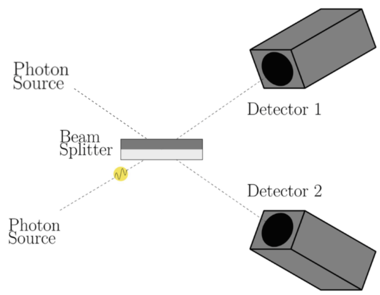

FIG. 1: Setup of the Hong-Ou-Mandel effect (Figure from [3])

We now introduce the mathematical description of this effect. Each input mode carries creation operators $a_1^\dagger$ and $a_2^\dagger$ and similarly each output mode carries creation operators $b_1^\dagger$ and $b_2^\dagger$. The action of the beam splitter is to transform $a_1^\dagger \to (b_1^\dagger + b_2^\dagger)/\sqrt{2}$ and $a_2^\dagger \to (b_1^\dagger - b_2^\dagger)/\sqrt{2}$.

Thus the initial two-photon state will be transformed as follows:

$$\begin{aligned}
|11\rangle &= a_1^\dagger a_2^\dagger |00\rangle \\
&\to \left(\frac{b_1^\dagger + b_2^\dagger}{\sqrt{2}}\right)\left(\frac{b_1^\dagger - b_2^\dagger}{\sqrt{2}}\right)|00\rangle \\
&= \frac{1}{2}\left(b_1^\dagger b_1^\dagger - b_1^\dagger b_2^\dagger + b_2^\dagger b_1^\dagger - b_2^\dagger b_2^\dagger\right)|00\rangle \\
&= \frac{1}{\sqrt{2}}\left(|20\rangle - |02\rangle\right),
\end{aligned}$$

which yields the measured probability distribution $P(11) = 0$ and $P(20) = P(02) = 1/2$. This result is a consequence of the symmetric nature of boson wave functions, which translates into the bosonic commutation relation $[b_i^\dagger, b_j^\dagger] = 0$. If the particles were fermions, the creation operators would satisfy the anti-commutation relations $\{b_i^\dagger, b_j^\dagger\} = 0$. So the final state would be $|11\rangle$, resulting in the probability distribution $P(11) = 1$ and $P(20) = P(02) = 0$.

A PATH TO QUANTUM ADVANTAGE

The Hong–Ou–Mandel effect demonstrates the significance of symmetry and antisymmetry in multi-photon interference. We will now extend this setup to a linear interferometer with N input and N output modes, where interference effects become even more dramatic. This interferometer is described by a unitary matrix $U \in U(N)$ that transforms $a_i^\dagger \to \sum_j U_{ij} b_j^\dagger$. For example, in the Hong–Ou–Mandel effect we had $N = 2$ and

$$U = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}.$$

Each unitary matrix U induces a unitary evolution $\rho(U)$ on the Hilbert space $\mathcal{H}$ of the photons [4]. Moreover, it can be shown that this map is an homomorphism $\rho(U_1U_2) = \rho(U_1)\rho(U_2)$. Hence, $U(N) \curvearrowright^{\rho} \mathcal{H}$ is a unitary representation of $U(N)$ on $\mathcal{H}$. To simplify the discussion, we focus on the subspace of at most N photons.

Since the number of photons is conserved, the representation decomposes as $\rho \sim \rho^{(0)} \oplus \rho^{(1)} \oplus \cdots \oplus \rho^{(N)}$, where $\rho^{(n)}$ is the representation on the subspace of n photons $\mathcal{F}_n$. For instance, in the vacuum subspace $\mathcal{F}_0 = \{|0\rangle\}$, ρ acts as the trivial representation. In the single photon subspace $\mathcal{F}_1 = \mathrm{Span}\{a_i^\dagger |0\rangle : 1 \le i \le N\}$, we find that

$$\rho^{(1)}(U) a_i^\dagger |0\rangle = U_{ij} b_j^\dagger |0\rangle,$$

allowing us to interpret U_{ij} as the amplitude for a single photon to travel from input mode i to output mode j.

To conclude, we derive two interesting results: the probability that all photons emerge in one output mode, $P(N0\ldots0)$, and the probability that they evenly distribute across output modes, $P(1\ldots1)$. After passing through the interferometer,

$$|1\ldots1\rangle = \prod_i a_i^\dagger |0\rangle \to \prod_i \left(\sum_j U_{ij} b_j^\dagger\right)|0\rangle.$$

Expanding this product, we obtain terms with all photons occupying a single mode $b_1^\dagger$ as

$$\prod_i U_{i1}(b_1^\dagger)^N |0\rangle + \cdots = \prod_i U_{i1}\sqrt{N!}\,|N\ldots0\rangle + \ldots$$

Thus, the probability that all photons appear in output mode 1 is $P(N0\ldots0) = N! \prod_i |U_{i1}|^2$. If the particles were distinguishable, the probability would be simply $|\prod_i U_{i1}|^2$, so we observe an enhancement factor of $N!$, reflecting the tendency for bosons to "bunch together" [5].

If we now collect terms where each $b_i^\dagger$ appears exactly once, we find

$$|1\ldots1\rangle \to \left(\sum_{\sigma \in S_N} \prod_i U_{i\sigma(i)}\right)|1\ldots1\rangle + \ldots.$$

Thus, the probability for an even distribution of photons across all modes is given by the squared modulus of the permanent of U, $P(1\ldots1) = |\mathrm{perm}(U)|^2$.

Unlike the determinant, the permanent is computationally intractable for large matrices, requiring $O(N2^N)$ operations classically compared to $O(N^3)$ for determinants [6]. This insight reveals a striking result: by observing the distribution of photons in this interferometer, we effectively compute a quantity — the permanent — that classical algorithms struggle to evaluate efficiently, demonstrating a form of quantum advantage [7].

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Appendix: Second Quantization Formalism

For completeness, we review the formalism of second quantization, which provides a convenient framework for tracking multiparticle states. We define the state $|n_1, n_2, \ldots, n_i, \ldots\rangle$, where n_i represents the number of particles occupying a single-particle basis state $|\psi_i\rangle$. The state $|0, 0, \ldots\rangle \equiv |0\rangle$, called the vacuum, contains no particles at all. The state $|0, 0, \ldots, n_i = 1, \ldots\rangle$ represents a single particle in the state $|\psi_i\rangle$.

Inspired by the ladder operators for the harmonic oscillator, we define the creation and annihilation operators:

$$a_i^\dagger |n_1, n_2, \ldots, n_i, \ldots\rangle = \sqrt{n_i + 1}\, |n_1, n_2, \ldots, n_i + 1, \ldots\rangle\,,$$
$$a_i |n_1, n_2, \ldots, n_i, \ldots\rangle = \sqrt{n_i}\, |n_1, n_2, \ldots, n_i - 1, \ldots\rangle\,.$$

To handle particle exchange symmetry, we impose commutation or anticommutation relations on the creation operators: $[a_i^\dagger, a_j^\dagger] = 0$ for bosons and $a_i^\dagger, a_j^\dagger = 0$ for fermions. These relations ensure that bosonic particles are symmetric under exchange, while fermionic particles are antisymmetric.