TÉCNICO LISBOA

Algebraic and Geometric Methods in Engineering and Physics

Rubik´s Cube Group

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I. Introduction

In 1974, Hungarian architecture professor Ernő Rubik created the famous three-dimensional puzzle known as the Rubik's Cube, also called the Magic Cube.

This puzzle is a cube with 6 faces, where each face is composed of small squares of a single colour, called cubies. There are various versions of the cube, with the most common being the 3x3x3 version, in which each face contains 9 cubies.

The cube can be scrambled by randomly rotating its six faces in succession in any direction. The goal is to restore the cube to its solved state, where each face displays only one uniform colour, that is, all cubies on each face have the same colour.

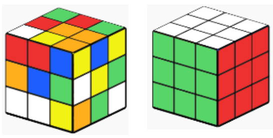

Figure 1 - Mixed and solved Rubik´s Cube

II. Notation

In 1981, David Singmaster introduced the notation that describes the series of turns of the faces needed to solve the cube. In this notation, it is assumed that the cube is on a flat surface and each turn of the face will be 90° clockwise:

- **F** denote the front face
- **R** denote the right face
- **U** denote the upward/top face
- **B** denote the back face
- **L** denote the left face
- **D** denote the downward/bottom face

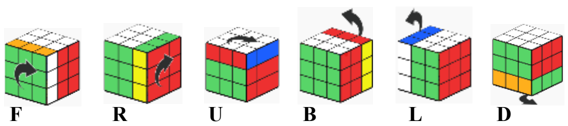

Figure 2 - Basic moves

The inverse of each move would be the 90° rotation of the face counter-clockwise and denoted F^{-1}, R^{-1},U^{-1}, B^{-1},L^{-1} and D^{-1}.

Regarding the cubies, it is possible to identify 3 types:

- **Center cubie** – 1 visible face
- **Corner cubie** – 3 visible faces
- **Edge cubie** – 2 visible faces

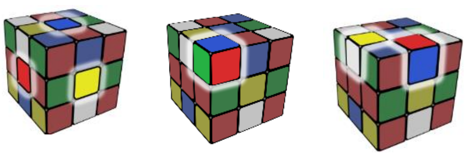

Figure 3 - Center, Corner and Edge cubies, respectively

III. Rubik´s Cube Group

It is possible to observe that, when applying a basic move to the cube, cubies are arranged and rearranged. So any position of the cube can be described as a permutation of the cubies, which leads to the possibility of describing the cube as a group of permutations.

Let $(G,*)$ be the Rubik's Cube Group such that:

- Group elements are all possible moves
- Two moves are considered the same if the final result is the same
- If G_1 and $G_2 \in G$, the G_1*G_2 is the move where G_1 is done first and G_2 is done after G_1

We prove that it is a group once it satisfies the 4 properties:

- **Closure:** if G_1 and G_2 are moves, G_1*G_2 is a move as well, so $G_1*G_2 \in G$
- **Associative:** let G_1, G_2, G_3 $\in G$. Both $G_1*(G_2*G_3)$ and $(G_1*G_2)*G_3$ mean to perform G_1 followed by G_2, followed by G_3. So $G_1*(G_2*G_3) = (G_1*G_2)*G_3 \in G$, $\forall\, G_1, G_2, G_3 \in G$
- **Identity:** there exists a neutral element $E \in G$, such that, $E*G_x = G_x*E = G_x$, $\forall G_x \in G$
  - ✓ $G_x*G_x*G_x*G_x = G_x^4 = E$, $\forall G_x \in G$

  <u>**Remark**</u>: E corresponds to not changing the cube at all
- **Inverse:** There exists a $G_X^{-1} \in G$, called the inverse of G_x, such that $G_X*G_X^{-1} = G_X^{-1}*G_x = E$
  - ✓ $G_X^{-1} = G_x*G_x*G_x$, $\forall G_x \in G$
  - ✓ $(G_1*G_2)^{-1}=G_2^{-1}*G_1^{-1}$,$\forall G_1, G_2 \in G$

<u>**Remark**</u>: Two sequences of moves are equivalents if the final result is the same, starting for the same initial condition, when an operation is executed, the cube is said symmetric to this operation.

This group is not commutative, that is, same sequence of moves in a different order can result in a different configuration. Therefore, it is a non-abelian group.

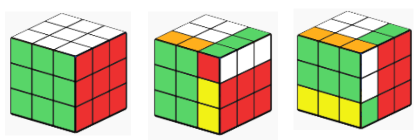

Figure 4 - Solved Cube, Configuration after FR, Configuration after RF, demonstrating the group's non-commutativity

We have 54 cubies that can be arranged and rearranged so the Rubik´s Cube Group is a subgroup of S_{54}, permutation group of 54 elements. Also, it is possible to describe any move as a sequence of basic moves. Therefore, basic moves are generators of the group, hence $G = \langle F,U,L,R,B,D \rangle \subset S_{54}$.

IV. Configurations

There are two different classifications of the Rubik´s Cube Group: the legal Rubik´s Cube Group and the illegal Rubik´s Cube Group. The illegal Rubik´s Cube Group allows to take the cube apart and rearrange the cubies without any restriction. On the contrary, in the legal Rubik´s Cube Group there are restrictions on the permutations. Only those that respect the movements of the cube without disassembling it are allowed, thus limiting the movements to the rotations of the faces.

As a result, in order to maintain the integrity of the cube, not all permutations of S_{54} can be possible on the Rubik´s Cube. The center cubie on each face is fixed, the edge cubie can only permute for another edge cubie and also, the corner cubie can only be sent for another corner cubie. Hence, G is only a subset of S_{54} and not isomorphic to the full permutation group.

Furthermore, each corner cubie can have, in the same position, three different orientations and each edge cubie can be two different orientations in the same position. This orientation is related to the colour of visible cubies.

Figure 5 - Orientations of corner cubes and edge cubies, respectively

So, the configuration of the Rubik´s Cube is described for the 4-tuple $(y,\sigma_{corner},x,\sigma_{edge})$ where:

- $y \in (\mathbb{Z}/3\mathbb{Z})^8$ represents the orientation of the corners;
- $\sigma_{corner} \in S_8$ represents the permutation of the corner cubies;
- $x \in (\mathbb{Z}/2\mathbb{Z})^{12}$ represents the orientation of the edges;
- $\sigma_{edge} \in S_{12}$ represents the permutation of the edge cubies.

However, not all of these configurations are valid. Valid configurations respect the first and the second fundamental theorem of cube theory.

- **First Fundamental Theorem of Cube Theory:**

Let $y \in (\mathbb{Z}/3\mathbb{Z})^8$, $\sigma_{corner} \in S_8$, $x \in (\mathbb{Z}/2\mathbb{Z})^{12}$ and $\sigma_{edge} \in S_{12}$. $(y,\sigma_{corner},x,\sigma_{edge})$ is a valid configuration if and only if:

1. <u>equal parity of permutations</u>: $\operatorname{sgn}(\sigma_{corner}) = \operatorname{sgn}(\sigma_{edge})$
2. <u>conservation of the total number of twists</u>: $y_1 + y_2 + \ldots + y_8 = 0 \pmod 3$
3. <u>conservation of the total number of flips</u>: $x_1 + x_2 + \ldots + x_{12} = 0 \pmod 2$

- **Second Fundamental Theorem of Cube Theory:**

An operation of the cube is possible if and only if the following are satisfied:

1. The total number of edge and corner cycles of even length is even
2. The number of corner cycles twisted right is equal to the number of corner cycles twisted left (up to modulo 3)
3. There is an even number of reorienting edge cycles

So, the Rubik´s Cube Group is given by $(G,*)$ where $G = \{(y,\sigma_{corner},x,\sigma_{edge}) \in ((\mathbb{Z}/3\mathbb{Z})^8, S_8, (\mathbb{Z}/2\mathbb{Z})^{12}, S_{12}) \mid$ the first and second fundamental theorem cube theory are satisfied$\}$.

V. Order of the Group

Now, knowing which configurations are valid, we can determine the total number of these configurations, in other words, we can compute the order of the group:

$$|G| = \frac{8! \times 3^8 \times 12! \times 2^{12}}{3 \times 2 \times 2} \cong 4{,}33 \times 10^{19}$$

Where $8! = |S_8|$, $12! = |S_{12}|$, $3^8 = |(\mathbb{Z}/3\mathbb{Z})^8|$, $2^{12} = |(\mathbb{Z}/2\mathbb{Z})^{12}|$

<u>Remark</u>: The demonym arises due to only $\frac{1}{3}$ of the permutations has the rotations of the corner cubies correct, only $\frac{1}{2}$ of the permutations has the same edge-flipping orientations as the original cube, and only $\frac{1}{2}$ of these has the correct cubie-rearrangement parity(even)

VI. Curiosities

Cayley graphs are a powerful mathematical tool that allows us to visualize and analyse the structure of groups and subgroups intuitively. They provide a graphical representation of the relationships between group elements, where vertices represent the elements and edges indicate how the elements relate through the group operation. These representations are particularly useful for finite groups, making them interesting for the Rubik's Cube Group. Here is a simple example of the Cayley graph for the subgroup generated by the moves ρ = RR and $\phi = FF$:

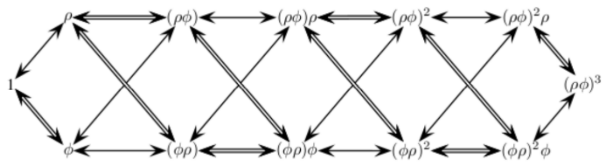

Figure 6 - Cayley Graph for subgroup generated by the moves ρ and ϕ

<u>Note:</u> Representing the complete Rubik's Cube Group is extremely complicated due to the high number of elements. The graph would have $|G| \cong 4{,}33 \times 10^{19}$ vertices.

These representations allow us to identify isomorphic groups because if two groups are called isomorphic, they have the same structure and thus the same Cayley graph.

An interesting aspect is that the diameter of the Cayley diagram of a Rubik's Cube is regarded as **God's Number**. God's number is the minimum number of moves required to solve any position of the Rubik's cube. In 2010, Tomas Rokicki proved that for the 3x3 Rubik's Cube, God's Number is 20.

VII. Conclusion

After all this study, we can conclude that group theory is extremely essential in the discussion of this puzzle, allowing us to discuss many propositions and properties of it. Thus, group theory simplifies the entire process of solving the puzzle, enabling us to obtain algorithms that guarantee a successful solution. All this analysis also demonstrates how abstract mathematics can be applied to real-world problems.

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