

Algebraic and Geometric Methods in Engineering and Physics

Homework 11

Due on December 11

1. Consider the maps $g, \omega : T_p\mathbb{R}^2 \times T_p\mathbb{R}^2 \rightarrow \mathbb{R}$ given by

$$g\left(v^1 \frac{\partial}{\partial x} + v^2 \frac{\partial}{\partial y}, w^1 \frac{\partial}{\partial x} + w^2 \frac{\partial}{\partial y}\right) = v^1 w^1 + v^2 w^2$$

and

$$\omega\left(v^1 \frac{\partial}{\partial x} + v^2 \frac{\partial}{\partial y}, w^1 \frac{\partial}{\partial x} + w^2 \frac{\partial}{\partial y}\right) = v^1 w^2 - v^2 w^1,$$

and the map $T : T_p\mathbb{R}^2 \times T_p^*\mathbb{R}^2 \rightarrow \mathbb{R}$ given by

$$T\left(v^1 \frac{\partial}{\partial x} + v^2 \frac{\partial}{\partial y}, p_1 dx + p_2 dy\right) = v^2(p_1 - p_2).$$

- (a) Show that g and ω are covariant 2-tensors on $T_p\mathbb{R}^2$.
- (b) Write g and ω in the basis $\{dx \otimes dx, dx \otimes dy, dy \otimes dx, dy \otimes dy\}$ of the covariant 2-tensors on $T_p\mathbb{R}^2$.
- (c) Show that T is a mixed tensor of type $(1, 1)$ on $T_p\mathbb{R}^2$.
- (d) Write T in the basis $\{dx \otimes \frac{\partial}{\partial x}, dx \otimes \frac{\partial}{\partial y}, dy \otimes \frac{\partial}{\partial x}, dy \otimes \frac{\partial}{\partial y}\}$ of the mixed tensors of type $(1, 1)$ on $T_p\mathbb{R}^2$.
- (e) If $v \in T_p(M)$ then we define the map $\iota(v)T : T_p^*M \rightarrow \mathbb{R}$ as $\iota(v)T(\alpha) = T(v, \alpha)$. Show that $\iota(v)T \in T_p^*M$, and write its expression in the basis $\{\frac{\partial}{\partial x}, \frac{\partial}{\partial y}\}$.