Algebraic and Geometric Methods in Engineering and Physics

2024/2025

2^{nd} Exam - 6 February 2025 - 10:30

Duration: 2 hours

(9/20) **1.** Consider the finite group $G = S_3 \times S_3$.

(a) Show that G is a nonabelian group of order 36.

(b) How many subgroups does G have? Of these, how many are nonabelian?

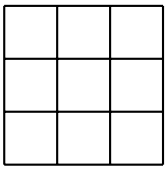
(c) Prove that $H = \{e, (123), (132)\} \times \{e\}$ is a normal subgroup of G, and identify the quotient group G/H.

(d) Show that if $\varphi : S_3 \to GL(n, \mathbb{C})$ is an irreducible representation of S_3 then each of the two maps $\varphi_1, \varphi_2 : G \to GL(n, \mathbb{C})$ given by $\varphi_1(\sigma, \tau) = \varphi(\sigma)$ and $\varphi_2(\sigma, \tau) = \varphi(\tau)$ are irreducible representations of G.

(e) List the conjugacy classes of G.

(f) How many irreducible representations does G have? What are their dimensions?

(3/20) **2.** Consider a square window divided into $3 \times 3 = 9$ equally sized square glasses as shown in the figure below. Each square glass can have one one of n possible colors. How many different windows are there, up to reorientation (i.e. rotations and/or reflections)? Check explicitly that your answer gives an integer for each choice of n.



(3/20) **3.** Let (N, d) be a metric space and let $f : M \to N$ be an injective function. Show that:

(a) $d_f : M \times M \to \mathbb{R}_0^+$ defined by $d_f(x, y) = d(f(x), f(y))$ for all $x, y \in M$ is a distance function on M.

(b) f is continuous for the topology defined on M by d_f.

(5/20) **4.** Consider the sets of matrices

$$SE(2) = \left\{ \begin{bmatrix} e^{i\alpha} & z \\ 0 & 1 \end{bmatrix} \in M_{2\times 2}(\mathbb{C}) : \alpha \in \mathbb{R}, z \in \mathbb{C} \right\}$$

and

$$\mathfrak{se}(2) = \left\{ \begin{bmatrix} i\alpha & z \\ 0 & 0 \end{bmatrix} \in M_{2\times 2}(\mathbb{C}) : \alpha \in \mathbb{R}, z \in \mathbb{C} \right\}.$$

Show that:

(a) $SE(2)$ is a Lie group of dimension 3.

(b) $\mathfrak{se}(2)$ is the Lie algebra of $SE(3)$.

(c) $\mathfrak{se}(2)$ is not simple.