Algebraic and Geometric Methods in Engineering and Physics 2024/2025

1st Exam - 23 January 2025 - 10:30
Duration: 2 hours

(9/20) **1.** Consider the finite group $D_5 = \{e, r, r^2, r^3, r^4, s, sr, sr^2, sr^3, sr^4\}$, where the generators s and r satisfy the relations $s^2 = r^5 = e$ and $r^{-1} = srs$.

(a) List all subgroups of D_5.

(b) Determine all 1-dimensional (irreducible) representations of D_5.

(c) Show that if $\alpha = \frac{2\pi}{5}$ or $\alpha = \frac{4\pi}{5}$ then

$$\varphi_s = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \qquad \varphi_r = \begin{pmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{pmatrix}$$

determines a 2-dimensional irreducible representation $D_5 \overset{\varphi}{\curvearrowright} \mathbb{C}^2$.

Hint: Recall that the matrix φ_r represents a rotation by an angle α.

(d) Compute the characters of all the irreducible representations above.

(e) How many equivalence classes of irreducible representations does D_5 have? What are their dimensions?

(f) List the conjugacy classes of D_5.

(2/20) **2.** Find all points with integer coordinates in the line $22x + 15y = 2$.

(3/20) **3.** Let $(M, \mathcal{T})$ be a compact topological space. Show that any closed set $F \subset M$ is also compact.

(6/20) **4.** Let $G \subset GL(n, \mathbb{C})$ be a matrix Lie group with Lie algebra $\mathfrak{g} \subset \mathfrak{gl}(n, \mathbb{C})$. The **fundamental** representation of G on $\mathbb{C}^n$ is the representation $G \overset{\varphi}{\curvearrowright} \mathbb{C}^n$ given by

$$\varphi_A(v) = Av.$$

The **dual representation** of G on $(\mathbb{C}^n)^* = \{\omega : \mathbb{C}^n \to \mathbb{C} \mid \omega \text{ is complex linear}\}$ is the representation $G \overset{\varphi^*}{\curvearrowright} (\mathbb{C}^n)^*$ determined by

$$\varphi_A^*(\omega)(\varphi_A(v)) = \omega(v).$$

Show that:

(a) If we identify $(\mathbb{C}^n)^*$ with $\mathbb{C}^n$ by setting $\omega(v) = \omega^t v$ for all $\omega, v \in \mathbb{C}^n$ then we have $\varphi_A^*(\omega) = (A^{-1})^t \omega$.

(b) The dual representation of $SU(2)$ is equivalent to the fundamental representation of $SU(2)$.

(c) The dual representation of $SU(3)$ is **not** equivalent to the fundamental representation of $SU(3)$.

(d) If $X^V \in \mathfrak{X}(G)$ the left-invariant vector field associated to $V \in \mathfrak{g}$ then its flow is given by $\phi_t^{X^V}(A) = Ae^{Vt}$.