Algebraic and Geometric Methods in Engineering and Physics

Homework 9

Due on December 7

1. Consider the stiffness matrix

$$K = \begin{pmatrix} 2k_1 + k_2 & -k_1 & -k_2 & -k_1 \\ -k_1 & 2k_1 + k_2 & -k_1 & -k_2 \\ -k_2 & -k_1 & 2k_1 + k_2 & -k_1 \\ -k_1 & -k_2 & -k_1 & 2k_1 + k_2 \end{pmatrix}$$

and the mass matrix $M = mI$, where where k_1, k_2 and m are positive numbers.

(a) Show that these matrices define intertwiners for the representation $D_4 \overset{\psi}{\curvearrowright} \mathbb{C}^4$ of the group $D_4 \equiv \{e, r, r^2, r^3, s, sr, sr^2, sr^3\}$ defined by

$$\psi_r = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}, \qquad \psi_s = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}.$$

(b) Recall that $\psi \sim \varphi^{(1)} \oplus \varphi^{(3)} \oplus \varphi^{(5)}$, where the irreducible representations $\varphi^{(1)}$, $\varphi^{(3)}$ and $\varphi^{(5)}$ are defined by $\varphi_r^{(1)} = \varphi_s^{(1)} = 1$, $\varphi_r^{(3)} = -1$, $\varphi_s^{(3)} = 1$ and

$$\varphi_r^{(5)} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \qquad \varphi_s^{(5)} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

Find the orthogonal projectors P_1, P_3 and P_5 on the invariant subspaces corresponding to these irreducible representations.

(c) Obtain the vibration frequencies of the structure described by K and M, that is, the values of ω such that $\omega^2 M u = K u$ for some nonvanishing vector u. Is the structure stable?