# Algebraic and Geometric Methods in Engineering and Physics

## Homework 13

*Due on January 11*

1. Consider the $2 \times 2$ complex matrices

$$\mathbf{1} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \mathbf{i} = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix},$$

$$\mathbf{j} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad \mathbf{k} = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}.$$

   (a) Check that $\mathbf{i}^2 = \mathbf{j}^2 = \mathbf{k}^2 = \mathbf{ijk} = -\mathbf{1}$.

   (b) Show that $\mathfrak{su}_2 = \operatorname{span}_{\mathbb{R}}\{\mathbf{i}, \mathbf{j}, \mathbf{k}\}$, and conclude that $\mathfrak{su}_2$ is isomorphic to $\mathfrak{so}_3(\mathbb{R})$.

   (c) Consider the four-dimensional vector space $\mathbb{Q} = \operatorname{span}_{\mathbb{R}}\{\mathbf{1}, \mathbf{i}, \mathbf{j}, \mathbf{k}\}$ (also known as the **quaternion algebra**) with the inner product defined by taking $\{\mathbf{1}, \mathbf{i}, \mathbf{j}, \mathbf{k}\}$ to be an orthonormal basis. Show that

$$SU_2 = \{\mathbf{q} \in \mathbb{Q} : \|\mathbf{q}\| = 1\},$$

   and conclude that $SU_2$ is simply connected.

   (d) Show that if $\mathbf{n} \in \mathfrak{su}_2 \cap SU_2$ then

$$\exp\left(\frac{\theta}{2}\mathbf{n}\right) = \cos\left(\frac{\theta}{2}\right)\mathbf{1} + \sin\left(\frac{\theta}{2}\right)\mathbf{n} \in SU_2.$$

   (**Hint:** Check that $\mathbf{n}^2 = -\mathbf{1}$).

   (e) Show that if $\mathbf{n} \in \mathfrak{su}_2 \cap SU_2$ then the map

$$\begin{aligned} \mathfrak{su}_2 &\to \mathfrak{su}_2 \\ \mathbf{v} &\mapsto \exp\left(\frac{\theta}{2}\mathbf{n}\right)\mathbf{v}\exp\left(-\frac{\theta}{2}\mathbf{n}\right) \end{aligned}$$

   is a rotation by an angle $\theta$ about the axis defined by $\mathbf{n}$.
   (**Hint:** Start by proving that there exists an orthonormal basis $\{\mathbf{l}, \mathbf{m}, \mathbf{n}\}$ of $\mathfrak{su}_2$ satisfying $\mathbf{l}^2 = \mathbf{m}^2 = \mathbf{n}^2 = \mathbf{lmn} = -\mathbf{1}$).

   (f) Show that there exists a surjective homomorphism $\Phi : SU_2 \to SO_3(\mathbb{R})$ What is the kernel of $\Phi$?