Algebraic and Geometric Methods in Engineering and Physics 2022/2023

2nd Exam - 9 February 2023 - 15:30
Duration: 2 hours

(12/20) **1.** Recall that the complex matrices

$$\mathbf{1} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \mathbf{i} = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}, \quad \mathbf{j} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad \mathbf{k} = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}$$

satisfy the quaternionic relations $\mathbf{i}^2 = \mathbf{j}^2 = \mathbf{k}^2 = \mathbf{ijk} = -\mathbf{1}$.

(a) Show that $G = \{\mathbf{1}, -\mathbf{1}, \mathbf{i}, -\mathbf{i}, \mathbf{j}, -\mathbf{j}, \mathbf{k}, -\mathbf{k}\}$ is a group under matrix multiplication.

(b) Show that $Z = \{\mathbf{1}, -\mathbf{1}\}$ is a normal subgroup, and identify the quotient group G/Z.

(c) Is G isomorphic to the dihedral group D_4?

(d) G acts naturally on $\mathbb{C}^2$ by matrix multiplication. Show that this action is an irreducible representation of G.

(e) How many irreducible representations does G have? What are their dimensions?

(f) G also acts naturally on $\mathrm{Mat}_2(\mathbb{C}) = \mathrm{span}_{\mathbb{C}}\{\mathbf{1}, \mathbf{i}, \mathbf{j}, \mathbf{k}\} \cong \mathbb{C}^4$ by matrix multiplication. Determine the decomposition of this representation into irreducible representations.

(4/20) **2.** Suppose that 3 identical decks of 52 cards are combined into a big deck. How many different 3 card hands can be dealt out of the big deck?

(2/20) **3.** Consider the family of subsets of $\mathbb{C}$ given by

$$\mathcal{T} = \{B_r(0) \, : \, r > 0\} \cup \{\varnothing, \mathbb{C}\}.$$

(a) Show that $\mathcal{T}$ is a topology on $\mathbb{C}$. Is this topology Hausdorff?

(b) Prove that any compact set for this topology must be bounded. Does it have to be closed with respect to the usual topology in $\mathbb{C}$?

(2/20) **4.** Consider the set

$$S^1 = \{z \in \mathbb{C} \, : \, |z| = 1\}.$$

(a) Prove that S^1 is a Lie group (for the usual multiplication of complex numbers).

(b) Determine all Lie group homomorphisms $\Phi : \mathbb{R} \to S^1$ (where the group operation in $\mathbb{R}$ is the sum).