Algebraic and Geometric Methods in Engineering and Physics

2022/2023

1st Exam - 26 January 2023 - 15:30
Duration: 2 hours

(9/20) **1.** Consider the set

$$G = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{Mat}_2(\mathbb{Z}_2) \,:\, ad - bc = 1 \right\}$$

with the operation of matrix multiplication.

(a) Show that G is a nonabelian group of order 6.

(b) Consider the action of G on $\mathbb{Z}_2 \times \mathbb{Z}_2$ by matrix multiplication. Determine wether this action is effective, free and/or transitive.

(c) By considering the restriction of this action to the set $\{v_1, v_2, v_3\}$, where

$$v_1 = (1, 0)\,, \quad v_2 = (0, 1)\,, \quad v_3 = (1, 1)\,,$$

prove that G is isomorphic to S_3.

(4/20) **2.** Recall that the dihedral group D_n is the finite group of order $2n$ determined by two generators r and s satisfying the relations

$$r^n = e\,, \quad s^2 = e\,, \quad rs = sr^{-1}\,,$$

where e is the identity.

(a) Prove that D_n has exactly two non-equivalent 1-dimensional representations if n is odd, and four non-equivalent 1-dimensional representations if n is even.

(b) How many non-equivalent irreducible representations does D_5 have? What are their dimensions?

(3/20) **3.** Consider the topological spaces

$$S^1 = \{z \in \mathbb{C} \,:\, |z| = 1\} \qquad \text{and} \qquad I = [0,1]\,,$$

both equiped with the usual (subspace) topology.

(a) Show that $S^1 \setminus \{1\}$ is homeomorphic to $\mathbb{R}$.

(b) Prove that $I \setminus \{\frac{1}{2}\}$ is disconnected.

(c) Conclude that S^1 and I are not homeomorphic.

(4/20) **4.** Consider the Lie algebra

$$\mathfrak{g} = \left\{ \begin{pmatrix} a & b \\ 0 & 0 \end{pmatrix} \in \mathrm{Mat}_2(\mathbb{C}) \,:\, a, b \in \mathbb{C} \right\}$$

where the Lie bracket is the commutator

(a) Is this Lie algebra simple?

(b) Find a Lie group G such that $\mathfrak{g}$ is its Lie algebra.

(c) Prove that there are only two complex Lie algebras of dimension 2 (up to isomorphism).