The Rubik's Cube Group

Algebraic and Geometric Methods in Engineering and Physics 2020/2021 Seminar

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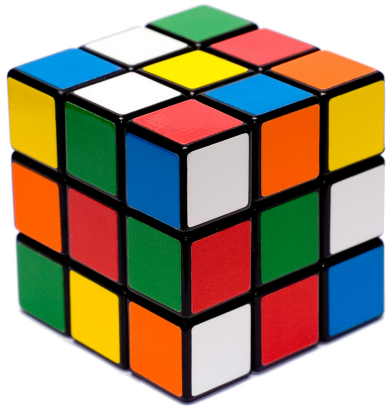
What is the Rubik's Cube?

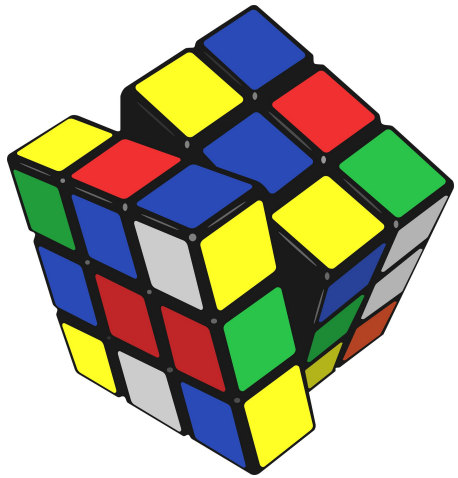
Figure: A Rubik's Cube



- The Rubik was invented by Hungarian architect Ernö Rubik, in 1974.
- It is made of 3x3x3 small cubes, called "cubies".

Manipulating the Rubik's Cube

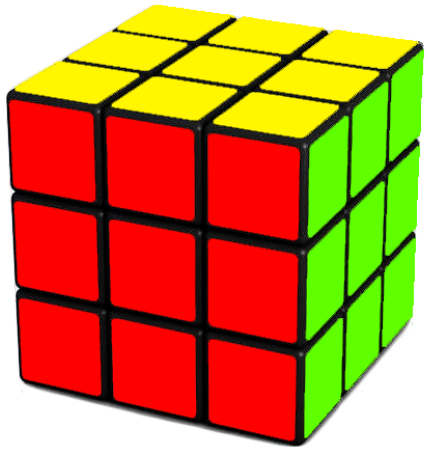
Figure: Rotating a face



- The cube can me manipulated by rotating the faces of the cube.
- The Rubik's cube is a puzzle: at the end, the 9 small faces of each face must all be the same color.

The Rubik's Cube as a Puzzle

Figure: The goal



The Goals of this Seminar

To explore the applications of the Group Theory to the Rubik's Cube:

- To give conditions to decide whether a configurations is valid or not.
- To estimate the number of valid configurations.

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Naming faces

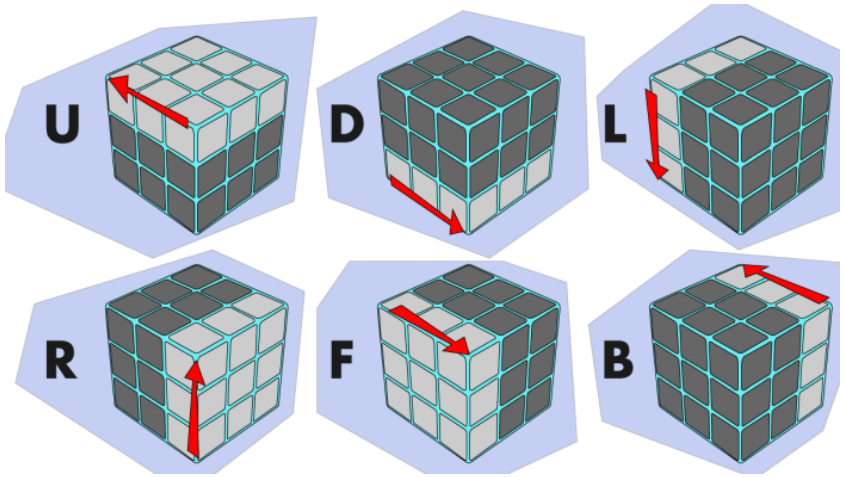
We are going to use the Singmaster Notation to describe faces:

- U denotes the upward (top) face.
- F denotes the front face.
- L denotes the left face.
- R denotes the right face.
- B denotes the back face.
- D denotes the downward (bottom) face.

Naming face moves

- The previous notation is also used to describe face moves (ex. F describes the rotation clockwise by $90^{\underline{o}}$, of the front face).
- These are the basic moves. All the other moves are sequences of the basic moves.

Figure: Basic face moves



Naming cubies

- Corner cubies are the cubies that have 3 visible faces.

Figure: Corner cubies

Naming cubies

- Edge cubies are the cubies that have 2 visible faces.

Figure: Edge cubies

Naming cubies

- Center cubies are the cubies that have 1 visible face.

Figure: Center cubies

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Making the Rubik's Cube into a Group

$(\mathbb{G}, *)$ is the Rubik's Cube Group.

- $\mathbb{G}$ is the set of all possible moves.
- Two moves are said to be the same if they result in the same configuration. For example, rotate the left face clockwise by $90º$ is the same as rotate the left face counterclockwise by $180º$.
- $M_1 * M_2$ is the move where you first do M_1 and then M_2.
- We can represent any cube configuration by detailing the sequence of moves from the start position to the that position.

Why is $(\mathbb{G}, *)$ a Group?

Let's check the four properties.

- (Closure) If $M_1, M_2 \in \mathbb{G}$, then M_1, M_2 are moves, and so $M_1 * M_2$ is a move as well, so $M_1 * M_2 \in \mathbb{G}$.
- (Identity) The element e is making no move at all. Then,

$$\forall M \in \mathbb{G}, M * e = e * M = M$$

- (Inverse) The inverse of a basic move $M \in \mathbb{G}$ is $M * M * M$; the inverse of the move $M * N$ is $N^{-1} * M^{-1}$.
- (Associativity) Let $X, Y, Z \in \mathbb{G}$. Both $X(YZ)$ and $(XY)Z$ mean making X, followed by Y, followed by Z. So, we have

$$\forall X, Y, Z \in \mathbb{G} : X(YZ) = (XY)Z$$

Generators of $\mathbb{G}$

Remark

All moves are sequences of one or more basic moves, so we have:

$$\mathbb{G} =< L, R, B, F, U, D >$$

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How many configurations?

- We can think of each element of $\mathbb{G}$ as a possible configuration of the Rubik's Cube. Each move is a permutation of 54 small faces. So

$$|\mathbb{G}| \leq |S_{54}|$$

- But each one of the 6 center cubies always stays on the same position. So

$$|\mathbb{G}| \leq |S_{48}|$$

- And so on... The goal of this seminar is to point out conditions for a configuration to be valid, so that we can compute $|\mathbb{G}|$

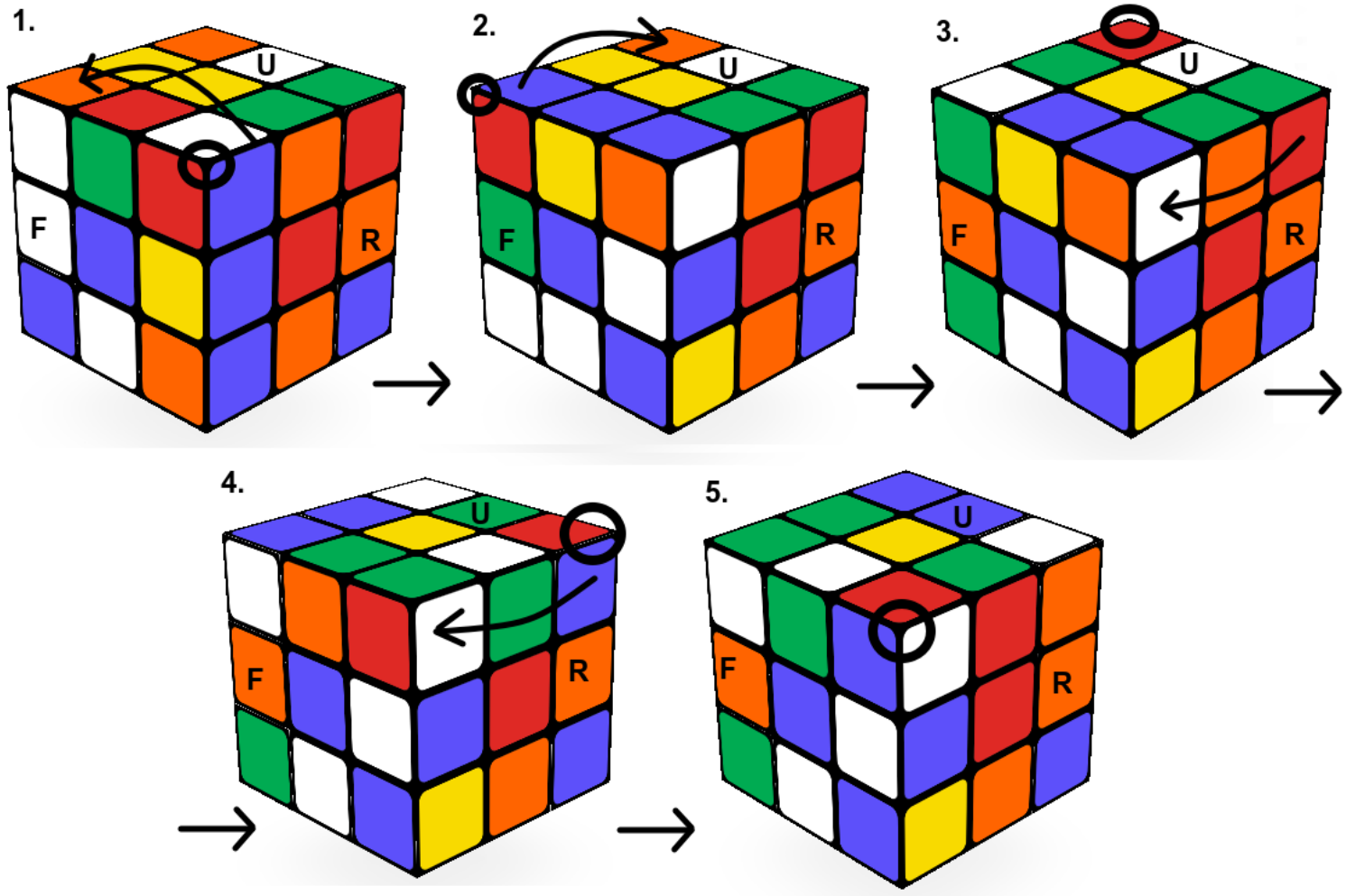
Valid permutations of the Rubik's Cube

A **valid** configuration is a configuration that can obtain from the solved one, by doing moves.

- Any valid permutation leaves the 6 center cubies in the same position.
- Any valid permutation sends the 8 corner cubies to corner positions.
- Any valid permutation sends the 12 edge cubies to edge positions.
- Each corner cubie can be in the same position in 3 different ways (3 different orientations).
- Each edge cubie can be in the same position in 2 different ways (2 different orientations).

Orientations of corner cubies

Figure: The same corner cubie, in the same position (1 and 5), but with different orientations. There are 3 possible orientations.



Orientations of edge cubies

Following the same reasoning as we have done for corner cubies, there are two different orientations for each edge cubie.

Configuration of the Rubik's Cube

So we have that each configuration of the Rubik's cube is specified by:

- A permutation of the corner cubies $\sigma \in S_8$.
- A permutation of the edge cubies $\tau \in S_{12}$.
- The orientation of the corner cubies $x \in (\mathbb{Z}/3\mathbb{Z})^8$.
- The orientation of the edge cubies $y \in (\mathbb{Z}/2\mathbb{Z})^{12}$.

Remark

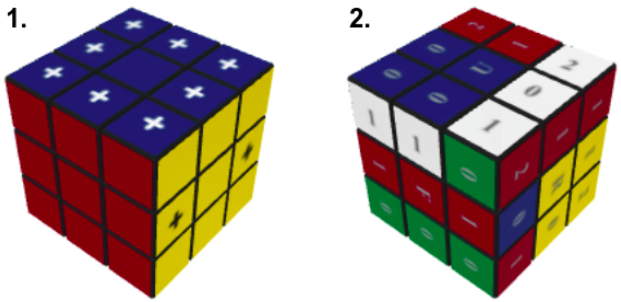
The configuration of the Rubik's Cube is the tuple

$$(\sigma, \tau, x, y) \in S_8 \times S_{12} \times (\mathbb{Z}/3\mathbb{Z})^8 \times (\mathbb{Z}/2\mathbb{Z})^{12}$$

Determining the components of the corner tuple

- For each corner **cubie**, we write the numbers 0, 1, 2 on the cubie faces (one number per face, without repeating). (See 2.)
- We write a '+' mark in each corner **position**. Each position has exactly one face with the '+' mark. (See 1.)
- These marks are assigned to positions, rather than cubies, so they never move. The numbers are assigned to cubies, then move.
- Each entry $x_i \in x$ is the number of the face of the ith corner cubie that corresponds to face with the '+' mark.

Figure: Marks



Determining the components of the edge tuple

We use the same reasoning as we used for corner cubies, but the marks are on the edge positions, and the numbers on the edge cubies are 0 and 1.

The Fundamental Theorem of the Cube Theory

However, not all $(\sigma, \tau, x, y) \in S_8 \times S_{12} \times (\mathbb{Z}/3\mathbb{Z})^8 \times (\mathbb{Z}/2\mathbb{Z})^{12}$ are valid configurations of the cube.

Theorem (Theorem 20.2.1 from [4])

Let $(\sigma, \tau, x, y) \in S_8 \times S_{12} \times (\mathbb{Z}/3\mathbb{Z})^8 \times (\mathbb{Z}/2\mathbb{Z})^{12}$. (σ, τ, x, y) is a valid configuration of the Rubik's Cube if and only if:

- (a) $sgn(\sigma) = sng(\tau)$
- (b) $x_1 + x_2 + ... + x_8 \equiv 0 \mod 3$
- (c) $y_1 + y_2 + ... + y_{12} \equiv 0 \mod 2$

Proof of the Fundamental Theorem of the Cube Theory I

- ($\Longrightarrow$) Let $M \in \mathbb{G}$. Then, $M = M_1 * M_2 * ... * M_k$, $M_i \in \{U, D, L, R, F, B\}$.
  - (a)

Basic move	σ	τ
R	(2 6 7 3)	(2 6 10 7)
L	(1 4 8 5)	(4 8 12 5)
U	(1 2 3 4)	(1 2 3 4)
D	(5 8 7 6)	(9 12 11 10)
F	(3 7 8 4)	(3 7 11 8)
B	(1 5 6 2)	(1 5 9 6)

All permutations (σ and τ of the basic moves) are 4-cycles, so for each M_i, we have:

$$sgn(\tau) = -1 = sgn(\sigma)$$

So, if $\sigma(M)$ denotes the corner cubies permutation of the move M,

$$sgn(\sigma(M)) = \prod_{i=1}^{i=k} sgn(\sigma(M_i)) = \prod_{i=1}^{i=k} sgn(\tau(M_i)) = sgn(\tau(M))$$

Proof of the Fundamental Theorem of the Cube Theory II

- (b) and (c)

Figure: How each move change orientations

M	x' and y'
D	$(x_1, x_2, x_3, x_4, x_8, x_5, x_6, x_7)$
	$(y_1, y_2, y_3, y_4, y_5, y_6, y_7, y_8, y_{10}, y_{11}, y_{12}, y_9)$
U	$(x_2, x_3, x_4, x_1, x_5, x_6, x_7, x_8)$
	$(y_4, y_1, y_2, y_3, y_5, y_6, y_7, y_8, y_9, y_{10}, y_{11}, y_{12})$
R	$(x_1, x_7 + 1, x_2 + 2, x_4, x_5, x_6, x_8 + 2, x_3 + 1)$
	$(y_1, y_7, y_3, y_4, y_5, y_2, y_{10}, y_8, y_9, y_6, y_{11}, y_{12})$
L	$(x_4 + 2, x_2, x_3, x_5 + 1, x_6 + 2, x_1 + 1, x_7, x_8)$
	$(y_1, y_2, y_3, y_5, y_{12}, y_6, y_7, y_4, y_9, y_{10}, y_{11}, y_8)$
F	$(x_6 + 1, x_1 + 2, x_3, x_4, x_5, x_7 + 2, x_2 + 1, x_8)$
	$(y_1, y_2, y_8 + 1, y_4, y_5, y_6, y_3 + 1, y_{11} + 1, y_9, y_{10}, y_7 + 1, y_{12})$
B	$(x_1, x_2, x_8 + 1, x_3 + 2, x_4 + 1, x_6, x_7, x_5 + 2)$
	$(y_6 + 1, y_2, y_3, y_4, y_1 + 1, y_9 + 1, y_7, y_8, y_5 + 1, y_{10}, y_{11}, y_{12})$

- ($\Longleftarrow$) See [1].

Applications of the Fundamental Theorem of the Cube Theory: $|\mathbb{G}|$

Corollary (Corollary 20.2.3 from [4])

We have

$$|\mathbb{G}| = \frac{2^{12} \cdot 3^8 \cdot 8! \cdot 12!}{12} \approx 4 \times 10^9$$

Applications of the Fundamental Theorem of the Cube Theory: $|\mathbb{G}|$

Proof.

- $|S_8 \times S_{12} \times (\mathbb{Z}/3\mathbb{Z})^8 \times (\mathbb{Z}/2\mathbb{Z})^{12}| = 8! \cdot 12! \cdot 3^8 \cdot 2^{12}$ is the total number of configurations of the Rubik's Cube.
- But by the last theorem, only $\frac{1}{12}$ are valid;
  - reduced by $\frac{1}{2}$ by (a) since there are as many even permutations as there are odd ones
  - reduced by $\frac{1}{3}$ by (b) since the orientation of 7 corner cubies can be arbitrarily chosen and this would determine the orientation of the 8th
  - reduced by $\frac{1}{2}$ by (c) since the orientation of11edge cubies can be arbitrarily chosen andthis would determine the orientation of the 12th

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God's number

- God's number is the maximum number of moves needed to solve each (valid) configuration of the Rubik's Cube.
- In 2010, it was proved that the number is 20.
- The team of reserchers programmed a computer to solve each one of the $|\mathbb{G}|$ (valid) configurations. See [5].

$\mathbb{G}$ is not abelian.

The order of the moves matters.

Figure: The sequences RU and UR applied to the cube in its initial state.

Initial position RU UR

$\mathbb{G}$ is not cyclic.

- The largest order of a move $M \in \mathbb{G}$ is 1260.

$$| < M > | \leq 1260 < 4 \cdot 10^9 = |\mathbb{G}|$$

- Example $R * U * U * D * D * D * B * D * D$.

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