Riemannian Geometry

2011/2012

1st Test - 24 January 2012 - 9:00

1. We define the **symplectic gradient** of a smooth function $F \in C^\infty(\mathbb{R}^2)$ to be the vector field $X_F \in \mathfrak{X}(\mathbb{R}^2)$ given in the usual Cartesian coordinates (x, y) by

$$X_F = \frac{\partial F}{\partial y}\frac{\partial}{\partial x} - \frac{\partial F}{\partial x}\frac{\partial}{\partial y}.$$

Show that:

(2/20) (a) If $a \in \mathbb{R}$ is a regular value of F then X_F is tangent to the submanifold $F^{-1}(a)$.

(2/20) (b) If F does not have critical points and

$$\lim_{(x,y)\to\infty} F(x, y) = +\infty$$

then X_F is complete.

(2/20) (c) $[X_F, X_G] = X_{\{F,G\}}$, where $\{F, G\} \in C^\infty(\mathbb{R}^2)$ is the function

$$\{F, G\} = X_F \cdot G = \frac{\partial F}{\partial y}\frac{\partial G}{\partial x} - \frac{\partial F}{\partial x}\frac{\partial G}{\partial y}.$$

(2/20) (d) If either F or G have a critical point and X_F, X_G are complete then their flows commute if and only if $\{F, G\} = 0$.

(2/20) (e) $\omega(X_F, Y) = dF(Y)$ for any $Y \in \mathfrak{X}(\mathbb{R}^2)$, where $\omega = dx \wedge dy$ is the standard volume form.

(3/20) (f) If X_F is complete with flow $\psi_t : \mathbb{R}^2 \to \mathbb{R}^2$ and $U \subset \mathbb{R}^2$ is a bounded open set such that $\overline{U}$ is a manifold with boundary then

$$\int_U \omega = \int_{\psi_t(U)} \omega.$$

(**Hint:** Use the Stokes Theorem on the three-dimensional manifold with boundary $M = \left\{(s, x, y) \mid s \in [0, t] \text{ and } (x, y) \in \psi_s\left(\overline{U}\right)\right\}$).

(3/20) (g) Check the result in (f) for the function $F(x, y) = x^2 + y^2$ by computing the flow of X_F explicitly.

(4/20) **2.** A smooth manifold M is called **contractible** if the identity map is smoothly homotopic to a constant map. Show that if M is compact and orientable (with $\dim M > 0$) then M is not contractible.

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2nd Test - 24 January 2012 - 9:00

1. Let $f : \mathbb{R} \to \mathbb{R}$ be a diffeomorphism, and consider the embedding $\varphi : \mathbb{R}^2 \to \mathbb{R}^3$ given by

$$\varphi(s,t) = (s\cos(t), s\sin(t), f(t))\,.$$

Show that:

(3/20) (a) The metric induced on $\mathbb{R}^2$ by this embedding and the Euclidean metric of $\mathbb{R}^3$ is

$$g = ds \otimes ds + R^2(s,t)\, dt \otimes dt,$$

where

$$R(s,t) = \sqrt{s^2 + (f'(t))^2}.$$

(3/20) (b) The nonvanishing connection forms associated to the orthonormal coframe

$$\omega^s = ds, \qquad \omega^t = R\, dt$$

are

$$\omega_s^t = -\omega_t^s = \frac{\partial R}{\partial s}\, dt.$$

(3/20) (c) The Gauss curvature is

$$K = -\frac{1}{R}\frac{\partial^2 R}{\partial s^2} = -\frac{(f'(t))^2}{(s^2 + (f'(t))^2)^2}.$$

(3/20) (d) The curve $s = 0$ and the curves of constant t are (images of) geodesics.

(3/20) (e) $\mathbb{R}^2$ with this induced metric is complete.

(4/20) **2.** Prove the **ultraparallel theorem**: given two non-intersecting geodesics γ_1, γ_2 of the hyperbolic plane ("parallel geodesics") there exists at most a third geodesic γ_3 (up to reparameterization) intersecting γ_1 and γ_2 orthogonally. Show by means of examples that γ_3 may or may not exist (that is, γ_1 and γ_2 may be **ultraparallel** or **asymptotic**).

(4/20) **3.** Let $N \subset \mathbb{R}^3$ be a 2-dimensional submanifold formed by straight lines through the origin (excluding the origin). Prove that N with the metric induced by the Euclidean metric of $\mathbb{R}^3$ is flat, that is, has zero Gauss curvature.

Cartan equations:

$$\begin{cases} d\omega^i = \sum_{j=1}^n \omega^j \wedge \omega_j^i \\ \Omega_j^i = d\omega_j^i - \sum_{k=1}^n \omega_j^k \wedge \omega_k^i \end{cases}$$