Riemannian Geometry
2011/2012
2nd Test - 15 December 2011 - 9:00

1. Let $f : I \to \mathbb{R}$ by a positive smooth function defined on some open interval $I \subset \mathbb{R}$, and consider the Riemannian metric g defined on $I \times \mathbb{R}$ by

$$g = \frac{1}{(f(x))^2}\left(dx \otimes dx + dy \otimes dy\right).$$

Show that:

(3/20) (a) The nonvanishing connection forms associated to the orthonormal coframe

$$\omega^x = \frac{1}{f(x)}dx, \qquad \omega^y = \frac{1}{f(x)}dy$$

are

$$\omega^x_y = -\omega^y_x = \frac{f'(x)}{f(x)}dy.$$

(3/20) (b) The Gauss curvature is

$$K = f(x)f''(x) - (f'(x))^2.$$

(3/20) (c) The curves of constant y are (images of) geodesics.

(3/20) (d) The choices $f(x) = 1$, $f(x) = x$, $f(x) = e^x$ and $f(x) = \cosh x$ lead to constant curvature manifolds. Which of these manifolds are complete, if we take I to be the interval where f is positive?

(4/20) **2.** Prove that if a 2-dimensional Riemannian manifold (M, g) with nonpositive Gauss curvature has a self-intersecting geodesic then the corresponding closed geodesic arc is not the boundary of a disk (and so M is not simply connected).

(4/20) **3.** Let $N \subset \mathbb{R}^3$ be a 2-dimensional submanifold, and $K : N \to \mathbb{R}$ its Gauss curvature. Prove that if N contains a straight line L then $K(p) \leq 0$ for all $p \in L$.

Cartan equations:

$$\begin{cases} d\omega^i = \sum_{j=1}^n \omega^j \wedge \omega^i_j \\ \Omega^j_i = d\omega^j_i - \sum_{k=1}^n \omega^k_i \wedge \omega^j_k \end{cases}$$

Curvature of a submanifold:

$$K^N(\Pi) = K^M(\Pi) + \frac{\langle B(X,X), B(Y,Y)\rangle - \|B(X,Y)\|^2}{\|X\|^2\|Y\|^2 - \langle X,Y\rangle^2}$$