Riemannian Geometry

2011/2012

1st Test - 3 November 2011 - 9:00

1. Consider the groups of matrices

$$GL(2,\mathbb{R}) = \left\{ \begin{pmatrix} x & y \\ z & w \end{pmatrix} \in \mathcal{M}_{2\times 2}(\mathbb{R}) \mid xw - yz \neq 0 \right\};$$
$$SL(2,\mathbb{R}) = \left\{ \begin{pmatrix} x & y \\ z & w \end{pmatrix} \in \mathcal{M}_{2\times 2}(\mathbb{R}) \mid xw - yz = 1 \right\}.$$

Show that:

(2.5/20) (a) $SL(2,\mathbb{R})$ is a 3-dimensional submanifold of $GL(2,\mathbb{R})$.

(2.5/20) (b) $T_I SL(2,\mathbb{R}) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathcal{M}_{2\times 2}(\mathbb{R}) \mid a + d = 0 \right\}.$

(2.5/20) (c) The left-invariant vector fields $X^V, X^W \in \mathfrak{X}(GL(2,\mathbb{R}))$ determined by

$$V = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \qquad W = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$$

are given in the above coordinates by

$$X^V = x\frac{\partial}{\partial y} + z\frac{\partial}{\partial w}, \qquad X^W = y\frac{\partial}{\partial x} + w\frac{\partial}{\partial z}.$$

(2.5/20) (d) $\left[X^V, X^W\right] = x\frac{\partial}{\partial x} - y\frac{\partial}{\partial y} + z\frac{\partial}{\partial z} - w\frac{\partial}{\partial w}.$

(2.5/20) (e) The flow of X^V at time $t \in \mathbb{R}$ is the map $\psi_t : GL(2,\mathbb{R}) \to GL(2,\mathbb{R})$ given by

$$\psi_t(x,y,z,w) = (x, y + xt, z, w + zt).$$

2. If M and N are compact, connected, oriented n-manifolds and $f : M \to N$ is a smooth map then the **degree** of f is the real number

$$\deg(f) = \frac{\int_M f^*\omega}{\int_N \omega}$$

where $\omega \in \Omega^n(N)$ is any n-form with nonzero integral (it can be shown that this number does not depend on the choice of ω).

(2.5/20) (a) Compute the degree of the antipodal map $f : S^n \to S^n$.

(2.5/20) (b) Prove that if n is even then any orientation-preserving diffeomorphism $f : S^n \to S^n$ has a fixed point, that is, a point $p \in S^n$ such that $f(p) = p$.

(2.5/20) (c) Using the fact that any closed 1-form on S^2 is exact, prove that the degree of any smooth map $f : S^2 \to S^1 \times S^1$ is zero.