Riemannian Geometry

Extra Homework 1

Optional, but must be handed in before the make up test for credit

Let (θ, φ) be the usual spherical coordinates on S^2, corresponding to the parameterization $\phi : (0, \pi) \times (-\pi, \pi) \to \mathbb{R}^3$ given by

$$\phi(\theta, \varphi) = (\sin\theta\cos\varphi, \sin\theta\sin\varphi, \cos\theta).$$

Let S^2_* be the open subset of S^2 covered by these coordinates (that is, S^2 minus the poles and a meridian). Show that:

1. The **cylindrical projection** $\pi_c : S^2_* \to (-1, 1) \times (-\pi, \pi)$, given in spherical coordinates by

   $$\pi_c(\theta, \varphi) = (\cos\theta, \varphi)$$

   preserves areas.

2. The **Mercator projection** $\pi_m : S^2_* \to (-\infty, \infty) \times (-\pi, \pi)$, given in spherical coordinates by

   $$\pi_m(\theta, \varphi) = \left(\log\left(\operatorname{cotg}\left(\frac{\theta}{2}\right)\right), \varphi\right)$$

   is conformal (that is, preserves angles).

3. The **stereographic projection** $\pi_s : S^2_* \to \mathbb{C}$, given by

   $$\pi_s(x, y, z) = \frac{x + iy}{1 - z},$$

   is written is spherical coordinates as

   $$\pi_s(\theta, \varphi) = \operatorname{cotg}\left(\frac{\theta}{2}\right) e^{i\varphi}.$$

4. The Mercator projection is the complex logarithm of the stereographic projection.

5. If the curve $c : \mathbb{R} \to S^2_*$ satisfies

   $$\pi_m(c(t)) = t\,(\cos\psi, \sin\psi) \qquad \mod (0, 2\pi)$$

   then c is a **loxodrome**, that is, a curve of constant bearing (and ψ is the constant angle between c and the north).

6. The stereographic projection maps the loxodrome c to the **logarithmic spiral**

   $$\pi_s(c(t)) = e^{t\cos\psi}\left(\cos(t\sin\psi), \sin(t\sin\psi)\right).$$

7. The total length of the loxodrome c is

   $$l(c) = \frac{\pi}{|\cos\psi|}.$$